Chapter 6: Force and Motion – II
(Newton’s 2nd with friction, drag & circular motion)

FRICTION
There are two types of frictional forces – static and kinetic friction. Symbolically they are written as

$$f_s = \text{Static frictional force} \quad \& \quad f_k = \text{kinetic frictional force}$$

**DEMO** Produce a friction curve using a heavy block with a spring scale

Consider a block that is being pulled to my right. Initially, I apply a small force and the block doesn’t move; the static frictional force is balancing out the force of my hand ($f_s = F_{\text{hand}}$). As I continue applying a larger force, the block still does not move. One must conclude that the static frictional force has increased in strength. In other words, the static frictional force is a VARIABLE FORCE (not a constant). At some point, my pulling force will be large enough to move the block, which indicates that the static frictional force reaches its maximum value at the so-called breakaway point. Just pass the breakaway point, the block easily slides along the table and there is a drop in the amount of pulling force that I have to apply. There is a change in frictional character: the reduced frictional force has changed from static to kinetic friction and it is easier to pull the block across the table. Plotting the force vs. time shows

**Interpretation** – this agrees with our intuition in everyday situations. Why? It is harder to start moving a couch from rest but once the couch is moving, it is easier to move. In the language of friction, when you first start pushing on the couch, the static friction force increases from $f_s$ to $f_{s,max}$, and the moment I exceed $f_{s,max}$, the couch starts moving and the frictional force switches from $f_{s,max}$ to $f_k$, which is smaller ($f_{s,max} > f_k$).

From a microscopic viewpoint, molecular irregularities of the two surfaces cause friction and is the electrical interaction between the molecules on each surface (and is beyond the scope of this course). If one was to look at polished stainless steel at the microscopic level, the ridges and valleys are apparent. We could naively model friction as moving across and over ridges and valleys that essentially catch and restrict travel.

**Limited list of properties of friction**
1. Frictional forces always oppose the direction of motion.

2. **Experimental facts about Friction**
   Even though friction depends on numerous parameters, experimental facts on friction show that at low speeds and very low heat transfer between the two surfaces, friction is proportional to the normal force: 
   
   $$\text{frictional force (f)} \propto \text{Normal force (N)} \iff f = \mu N$$
   
   where $\mu$ is called the **coefficient of friction**.
What does $f_{\text{static}} \geq f_{\text{kinetic}}$ mean?

To start the car moving, the engine delivers a torque that makes the wheels rotate. When a car is accelerating, the unbalanced force provided by the road is the frictional force according to Newton’s 3rd law. In other words, as the tires push backwards, the force of the road pushes forward such that $F_{\text{road}} = f_S$, not $f_k$. What does this mean?

- If the road is frictionless ($\mu_k = 0$), then the road is either icy or muddy, and the tires will merely spin in place. Technically, we say that the tires are moving (“sliding”) relative to the road.
- If there is friction ($\mu_S \neq 0$), the wheels will not slip. Technically, we say that the tire surface that is in contact with road is at rest relative to the road. Let me say this again, because it appears to not make sense. The tire is NOT moving (i.e., not sliding) relative to the road. Therefore, the friction between the road and the tire is static friction, NOT kinetic friction!

Examples. An image of a rolling tire shows that the top of the wheel the spokes are blurred, indicating that the spokes are moving fast. On the other hand, the bottom spokes are well defined and appear to be “stationary” relative to the ground. So I am claiming, which is correct, that the tire at the contact point to the ground is at REST. If the tire at the bottom is at rest, then the tire is not sliding but static in a “moving frame of reference.”

Indirect confirmation: Look at tire tracks in snow or mud, and you will find that there are clean tire tracks (not smudged). This indicates that the tires are not sliding but at rest relative to the ground ($v_{\text{bottom}} = 0$ at the contact point). Therefore, static friction is at work, NOT kinetic friction. Below are tire treads patterns in mud, sand, and snow.

Suppose we are in a car that is stuck in snow. If one accelerates too fast, the tires will slip (or slide) relative to the snow surface and it is kinetic friction at work, which has a smaller value. However, if the accelerator pedal is lightly pressed such that the
tires do not slip (or slide) relative to the snow, static friction is at work, which is larger and therefore, increases your chances of getting free.

On a similar note, your car brakes operate on a very similar premise. If one was to suddenly apply the brakes really hard and “lock your wheels” we mean that the tires will not roll but slide relative to the ground – kinetic friction $f_{\text{kinetic}}$ is at work here. On the other hand, Antilock Brake System (ABS) does not allow one to lock their brakes. Even though one may “slam on their brakes,” the tires continue rolling so that static friction is doing the work and the wheels provide maximal stopping force.

Remarks
Let’s be clear what this means when we use this equation $f = \mu N$. We are stating that friction only depends on the surface type (coefficient of friction) and the normal force. That is, friction does NOT depend on the following things:

1. **Does not depend on speed** since $f = \mu N$.
   A car skidding at low speeds (10 mph) has the same stopping force as a car skidding at high speeds (1000 mph)! Really? This is crazy since a car braking at 1000 mph will completely melt those tires off the car. So what we mean when we say $f = \mu N$ is that the speed of the car must be relatively low so that the heating of the tire is minimal. If this is the case, experiments show that friction does not depend on velocity.

   Aside: friction does depend on velocity. Have you ever noticed that when you are driving and are applying your brakes, one has to ease up on the brakes as you slow down because friction increases with decreasing speed (i.e., slowing down). On the other hand, as you speed up and apply the brakes, it is harder to stop the car since friction decreases as speed increases!

2. **Does not depend on surface area.**
   Friction does not depend on surface area since there is no area term in $f = \mu N$. What does this mean?
   - At low speeds, those extra wide tires you see on some cars provide no more friction than narrower tires. The wider tires simply spread the weight of the car over more surface area reducing heating and wear.
   - The friction between a truck and the ground is the SAME whether the truck has 4 tires or 18! (assuming the same normal force, of course) More tires spread the load over more ground area and reduce the pressure per tire. Interestingly enough, the stopping distance when the brakes are applied is NOT affected by the number of tires but the wear that tires experience is.

   Should you be bothered by these statements? I definitely am. Friction is much more complicated than $f = \mu N$, but if we look closer, it is way beyond the scope of this course, so we accept this simple equation for our problems.

The Normal Force on an Incline and Friction
Suppose a block is resting on an incline where there is friction between the surface and the block. (i) What is the role of the normal force in friction problems? (ii) At what critical angle does a block slide down the incline from rest?

Drawing a FBD and applying N2L, one gets
\[
\begin{align*}
\sum F_x &= mg\cos(90 - \theta) - f_{s,\text{max}} = 0 \\
\sum F_y &= N - mg\sin(90 - \theta) = 0 \\
\implies f_{s,\text{max}} &= \mu_s N = mg\sin\theta \\
\mu_s mg\sin\theta &= mg\cos\theta &\implies \mu_s = \tan\theta
\end{align*}
\]

**Remark**

1. At the break away point, it is the point where the x-component of the weight (mgx) exceeds the value of \(f_{s,\text{max}}\) (mgx > f_{s,\text{max}}).
2. The relationship between the coefficient of friction and the angle is mass independent. If two different blocks of different masses are on the incline, and angle of the incline is increased, both blocks to slide at the same critical angle value. This is a very difficult demo to do.

**DEMO** incline with two different blocks

**Check Questions**

A car is being driven up a steep hill at constant velocity.
- What pushes the car up the hill? Details required.
- Do the tires grip the road better on level ground or when going up the hill?
- How about coming down the hill instead?

**Example 6.1**

A loaded penguin sled weighing 80 N rests on an inclined plane at 20° to the horizontal. Between the sled and the plane, the coefficient of static friction is 0.25, and the coefficient of kinetic friction is 0.15. (a) What is the minimum magnitude of the force, parallel to the plane that will prevent the sled from slipping down the plane? (b) What is the minimum magnitude \(F\) that will start the sled moving up the plane? (c) What value of \(F\) is required to move the sled up the plane at constant velocity?

**Solution**

a. The sled is on the verge of sliding (but is still stationary) down the incline and we want to apply the minimum amount of force (\(F_{\text{min,1}}\)) to keep it from sliding down. If the sled is on the verge of sliding the frictional force is at its maximal value (\(f_{s,\max}\)) and will oppose this tendency; therefore, \(F_{\text{min,1}}\) and \(f_{s,\max}\) point in the same direction. Drawing out the FBD and applying N2L to the sled (m = 80/9.81 = 8.2 kg), we write

\[
\begin{align*}
\sum F_x &= F_{\text{min,1}} - mg\cos 70 + \mu_s N = 0 \\
\sum F_y &= N - mg\sin 70 = 0
\end{align*}
\]

Solving for the normal and substituting it into the x-equation, we can then solve for \(F_{\text{min,1}}\):

\[
\begin{align*}
F_{\text{min,1}} &= mg\cos 70 - \mu_s N \\
N &= mg\sin 70
\end{align*}
\]

\[
F_{\text{min,1}} = mg(\cos 70 - \mu_s \sin 70) = 80 \text{ N (cos 70} - 0.25 \sin 70) = 8.6 \text{ N} = F_{\text{min,1}}(\text{friction})
\]

What does this mean physically? If there was no friction, the minimum force \(F_{\text{min,1}}\) required would only have to balance out \((mg)\):

\[
F_{\text{min,1}}(\text{frictionless}) = mg\cos 70 = 27.5 \text{ N}
\]

Comparing the minimum values for frictionless vs. friction, we see that
Clearly, the minimum force to hold up the sled with friction is significantly smaller. In my rock climbing days, I relied heavily by making as much contact with the rock in certain situations to reduce the use of my arms to get this type of help from friction.

b. As soon as the sled wants to move upwards, the frictional forces immediately changes direction (remember the peaks in the image) and points downhill. Since the sled is on the verge of moving (but still stationary), the frictional force is at its maximal value again but this time opposes the force \( F_{\text{min},2} \). Drawing a FBD and applying \( \text{N2L} \) leads to

\[
\begin{align*}
\sum F_x &= F_{\text{min},2} - mg \cos 70 - f_{\text{s, max}} = 0 \\
\sum F_y &= N - mg \sin 70 = 0
\end{align*}
\]

Solving for the normal and \( F_{\text{min},2} \) as we did in part (a), we get

\[
F_{\text{min},2} = mg \cos 70 + \mu_s N = mg(\cos 70 + \mu_s \sin 70) = 46 \text{ N} = F_{\text{min},2}
\]

**What does this mean physically?**

Comparing the minimal values again for frictionless vs. friction, we see that

\[
F_{\text{min},2} \text{ (frictionless)} = 27.5 \text{ N} \quad \text{vs.} \quad F_{\text{min},2} \text{ (friction)} = 46 \text{ N}
\]

\[
\frac{46 - 27.5}{46} = 40\% \text{ harder}
\]

When comparing the two different minimums, friction can play a substantial role:

\[
F_{\text{min},1} = 8.6 \text{ N} \quad \text{vs.} \quad F_{\text{min},2} = 46 \text{ N}
\]

- \( F_{\text{min},1} \) is much smaller because friction reduces \((mg)_x\) and helps to reduce the required force to keep the block from moving.
- \( F_{\text{min},2} \) is much larger because friction helps \((mg)_x\) to oppose the force to move the block upwards.

a. The only way the block will move up the incline is if \( F_{\text{net},x} \neq 0 \), which occurs when \( F_{\text{net},x} > mg \cos 70 + f_{\text{s, max}} \). On the other hand, (ii) the only way the block slides down the incline is when \( mg \cos 70 > F_{\text{min},1} + f_{\text{s, max}} \). Since the block is first accelerated and is assumed to move at constant velocity, the frictional forces change from static to kinetic friction (similar to Mr. Hide changing to Dr. Jekel), and there is a decrease in the frictional force. This minimal force \( F_{\text{min},3} \) to keep the block moving at constant velocity up the incline is

\[
F_{\text{min},3} = mg \cos 70 + \mu_k N = 80 \text{ N} (\cos 70 + 0.15 \cdot \sin 70) = 39 \text{ N} = F_{\text{min},3}
\]

**What does this mean physically?**

The minimal forces to just start accelerating the block up the incline is larger than the minimal force to keep the block moving at constant velocity:

\[
F_{\text{min},2} \text{ (to accelerate)} \geq 46 \text{ N} \quad \text{vs.} \quad F_{\text{min},3} \text{ (constant velocity)} = 39 \text{ N}
\]

\[
f_{\text{static}} > f_{\text{kinetic}}
\]

One sees that it is easier to push an object once you “get it going.”

Let’s now look at a more challenging friction problem.
Example 6.2

The two blocks (with \( m_1 = 16 \text{ kg} \) and \( m_2 = 88 \text{ kg} \)) shown are not attached. The coefficient of static friction between the blocks is 0.38, but the surface beneath block-2 is frictionless. What is the minimum magnitude of the horizontal force \( F \) required to keep the smaller block from slipping down the larger block?

**Solution**

As block-1 is being pushed by force \( F \) into block-2. The contact between both blocks is \( F_{12} (= F_{21}) \), which is the normal force of block-2 on block-1 and the normal force used for the frictional force of block-1. If block-1 is on the verge of sliding down, then the static friction force is maximal \( (f_{s,1} = \mu_s F_{12}) \). Since block-1 is being held up by friction alone, the “strength” of the normal force \( F_{12} \) depends on how hard you apply the force \( F \) to block-1.

**DEMO** two blocks

Drawing the FBDs and apply N2L to both blocks,

\[
\begin{align*}
\Sigma F_{1,x} &= F - F_{12} = m_1a \\
\Sigma F_{1,y} &= f_{s,1} - m_1g = \mu_s F_{12} - m_1g = 0
\end{align*}
\]

\[
\begin{align*}
\Sigma F_{2,x} &= F_{12} = m_2a \\
\Sigma F_{2,y} &= N - m_2g = 0
\end{align*}
\]

So the N2L equations have setup three equations and three unknowns \((F, F_{12}, \text{ and } a)\) since \( \Sigma F_{2,y} \) is not relevant to solving for these unknowns. To solve for the minimum force \( F \), we see from equation \( \Sigma F_{1,x} \) that it depends on both \( F_{12} \) and \( a \).

\[
\Sigma F_{1,x} = F - F_{12} = m_1a \quad \rightarrow \quad F = F_{12} + m_1a
\]

So we have to determine \( F_{12} \) and acceleration from the other equations. We get \( F_{12} \), we use \( \Sigma F_{1,y} \), and the acceleration comes from equation \( \Sigma F_{2,y} \):

\[
\Sigma F_{1,y} = \mu_s F_{12} - m_1g = 0 \quad \rightarrow \quad F_{12} = \frac{m_1g}{\mu_s}
\]

\[
\Sigma F_{2,x} = F_{12} = m_2a \quad \rightarrow \quad a = \frac{F_{12}}{m_2}
\]

Substituting these two expressions into the \( F \)-equation:

\[
F = F_{12} + m_1a = \frac{m_1g}{\mu_s} + m_1 \cdot \frac{m_2 \mu_s}{m_2} = \frac{m_1g}{\mu_s} \left(1 + \frac{m_1}{m_2} \right) = \frac{m_1g}{\mu_s} \left(1 + \frac{16 \text{ kg}}{m_2 = 88 \text{ kg}} \right) = 488 \text{ N} = F
\]

6.6
CIRCULAR MOTION
In Chapter 2, we spoke of ways one could change velocity: either (i) speed change with constant direction or (ii) direction change with constant speed. Any change in velocity is acceleration and the net force causes this effect. Therefore, net forces can be placed into two categories: (i) forces that change the speed of the particle but not its direction and (ii) forces that change the direction of the particle but not its speed.

Properties of the centripetal force and acceleration
1. When circular motion occurs, the net force is called the centripetal force \( F_C \).
2. The centripetal force \( F_C \) is perpendicular to the direction of motion and always points towards the center of the circular motion. Once again, \( F_C \) is not in the direction of the velocity. The effect of \( F_C \) causes objects to turn in circles without alternating their speed. Only net forces parallel/antiparallel to the velocity will cause speeding up or slowing down. For this reason, in some textbooks \( F_C \) is called a deflection force since it only causes directional changes.

What is the Centripetal Force?
According to N2L,
\[
F_{\text{net}} = m\ddot{a} \quad \text{valid for } v = \text{constant} \\
F_{\text{net}} = F_{\text{centripetal}} = m\ddot{a}_C \quad \text{valid for } v \neq \text{constant}
\]

ii. The centripetal force is not a physical force – it is the net force. Very similar to what we talked about when an object is being accelerated as my hand pushes a block across the table. It is not my hand that is accelerating the block; it is the net force that is accelerating the block and equal to \( F_{\text{net}} = F_{\text{hand}} - f \). The centripetal force is something from the environment and is the component of the net force directed along the radial line that causes (or supplies) the centripetal acceleration.

DEMO Plate with cup
As I swing a cup vertically on a plate, what is the \( F_C \) required to cause circular motion of the cup?
\[
F_{C,\text{bottom}} = N - mg = ma_C; \quad F_{C,\text{top}} = N + mg = ma_C; \quad F_{C,\text{side}} = N = ma_C
\]

iii. Misconceptions about the centripetal force
The \( F_C \) is not a new kind of force and there is a misconception that an object moving in a circle has an outward force acting on it. So let’s analyze this situation and hopefully, clear this misconception once and for all. First of all, if you are in a car going around the circular curve, you are not in an inertial but in an accelerated frame of reference. Remember that Newton’s laws are only valid when the observer is in an inertial frame of reference (at rest or constant velocity). In the accelerated frame, the observer inside the accelerating car feels a “force pushing on them” making them move towards the door. To be explicitly clear, according to N2L there must be a net force acting on this person to make them move (or accelerate). Can anyone identify this force? NO! This force is a fictitious force and is residue of the accelerated frame of reference that the observer is in. This fictitious force is called a centrifugal force which means “center fleeing.” Note we can easily answer this question for an observer outside the accelerating car.
A person inside the car wants to continue moving along a straight line at constant velocity, and will continue until a net force forces this person to accelerate. As the car is forced to make a turn, the direction of the car door changes but the person inside the car continues at a constant velocity. Consequently, the driver runs right into the door, and the door stops their straight line motion. See – there is no force causing them to move towards the door.

Here is another example of an accelerating frame of reference residue

**DEMO** Swing a mass on a string

- When this mass is undergoing circular motion, it appears that an outward force is caused by the tension on my hand. However, according to Newton’s 3rd law, the hand-tension system forms an action-reaction pair. **There is no outward force!**
- If there was an outward force, we should see evidence in the following little experiment. Suppose I release a swing mass directly at her/his forehead; if there is an outward force then I should clock them on the forehead.

3. There are two associated quantities with circular motion: (i) period $T$ and (ii) angular frequency $\omega$. The period is the time for an object to make one complete revolution. If an object has a tangential velocity of $v$, then its speed is given by

$$v = \frac{\Delta x}{\Delta t} = \frac{2\pi r}{T}$$

However, connected to the period is its angular frequency, how often an object makes a full revolution ($2\pi$) in radians:

$$\omega = \frac{\Delta \theta}{\Delta t} = \frac{2\pi}{T} = \frac{v}{\frac{2\pi r}{v}} = \frac{2\pi}{r}$$

4. Deriving the centripetal acceleration

The centripetal force $F_C$ causes the centripetal acceleration $a_C$. I will derive $a_C$ both physically or mathematically for uniform circular motion.

**Physical Approach**

Imagine an object steadily traversing a circle of radius $r$ centered on the origin. Its position can be represented by a vector of constant length that changes angle. The total distance covered in one cycle is $2\pi r$. This is also the accumulated amount by which position has changed.

Now consider the velocity vector of this object: it can also be represented by a vector of constant length that steadily changes direction. This vector has length $v$, so the accumulated change in velocity is $2\pi v$. A more physical approach is that I have assumed that the speed of the particle is constant. As it goes around the circle its direction of motion changes. That means the velocity vector keeps the same length but turns through a whole circle. The tip of the velocity vector describes a circle of radius $v$, so the distance that the tip moved was $2\pi v$. The magnitude of acceleration is the change in velocity over the elapsed time, which we can write as

$$a_C = \frac{\Delta v}{\Delta t} = \frac{2\pi v}{T} = \frac{2\pi v}{2\pi r/v} = \frac{v^2}{r} = a_C$$

**Mathematical approach**

Assume that the length of the displacement vector has constant length but that the angle is changing as a function of time at a constant rate (we call this uniform circular motion). The displacement vector is broken into components and written as

$$\vec{r}(t) = r \cos \theta(t) \hat{i} + r \sin \theta(t) \hat{j}$$
To determine the acceleration, we take a two time derivatives of the displacement vector. The first derivative gives us the velocity:

\[ \ddot{r} = \frac{d^2\hat{r}(t)}{dt^2} = r \left[ -\sin(\theta(t)) \frac{d\theta}{dt} \hat{i} + \cos(\theta(t)) \frac{d\theta}{dt} \hat{j} \right] \]

The angular frequency is defined as \( d\theta/dt \equiv \omega = \text{constant} \). Next we take the derivative of the velocity and this gets us the acceleration:

\[ \ddot{a}_c = \frac{d\ddot{r}(t)}{dt} = \omega^2 \left[ \cos(\theta(t)) \frac{d\theta}{dt} \hat{i} + \sin(\theta(t)) \frac{d\theta}{dt} \hat{j} \right] \]

The magnitude of the acceleration is

\[ |\ddot{a}_c| = \left| -\omega^2 \ddot{r}(t) \right| \to \ddot{a}_c = \frac{v^2}{r} \]

5. Nonuniform Circular Motion \( \equiv \) changing both speed and direction:

\[ \ddot{a} = (a_t, a_c) \text{ where } a_t = \frac{\Delta v}{\Delta t}, \quad a_c = \frac{v^2}{r} \rightarrow \left\{ \begin{array}{l}
\dot{a} = \sqrt{a_t^2 + a_c^2} \\
\theta = \tan^{-1}(a_c/a_t)
\end{array} \right. \]

Questions

b. Why does mud fly off a rapidly turning automobile tire?

c. A coin is put on a phonograph turntable. The motor is started but, before the final speed of rotation is reached, the coin flies off. Explain why.

Example 6.3

A conical pendulum has a bob (0.040 kg) that moves in a horizontal circle at constant speed. What are (a) the tension in the string and (b) the period of the motion?

Solution

a. Draw a FBD and sum the components:

So N2L gives us two equations and two unknowns (F and v). To solve for tension F, we need to first determine \( \theta \). The angle \( \theta \) that the cord makes with the horizontal is determine by the triangle:

\[ \cos \theta = \frac{r}{L} \rightarrow \theta = \cos^{-1}(0.15/0.90) = 80^\circ \]

Now we can solve for tension F; from the y-equation we get

\[ \sum F_y = F \sin \theta - mg = 0 \quad \rightarrow \quad F = \frac{mg}{\sin \theta} = \frac{0.40 \ N}{\sin 80^\circ} = 0.40 \ N = F \]

b. The velocity is related to the period of the orbit, and the horizontal component of that tension causes the centripetal acceleration. Solving for the velocity from the x-equation,

\[ \sum F_x = F \cos(80) = ma_c = mv^2/r \quad \rightarrow \quad v = \sqrt{\frac{rF \cos(80)}{m}} = 0.49 \ m/s \]

By definition of the speed, the period of revolution is
Let's consider now an example that is similar to the elevator problem we discussed in Chapter 5. Quick reminder:

\[ F = mg + ma \rightarrow \text{"feel heavier"} \]

\[ F = mg - ma \rightarrow \text{"feel lighter"} \]

Newton’s Condition on Circular Motion

As this mass is whirling in a vertical circle, what is the centripetal force acting on this mass? N2L tells us

\[ T_{\text{top}} = m \left( \frac{v^2}{R} - g \right) \quad \text{(smallest tension)} \]

\[ T_{\text{bottom}} = m \left( \frac{v^2}{R} + g \right) \quad \text{(greatest tension)} \]

Similar to the elevator problem, the mass at the top of the loop experiences the smallest tension whereas at the bottom it experiences the greatest tension. If you look at the equation for the top of the loop,

\[ T_{\text{top}} = m \left( \frac{v^2}{R} - g \right) = m(a_c - g) \]

there is a limit to when this equation makes sense. Let’s physically try to understand this deeper.

As the mass heads from the bottom of the loop towards the top, there are two accelerations to speak of, \( a_c \) vs. \( g \). The role of the acceleration of gravity is to decrease the velocity of the mass and therefore, decrease the centripetal acceleration since it depends on velocity. However, the role of the centripetal acceleration is to keep the tension taut. If the velocity is slowed down too much, there is a critical point where the centripetal acceleration will not be able to keep the tension taut. If

\[ a_c < g \rightarrow T_{\text{top}} = m(a_c - g) < 0 \rightarrow \text{no tension on mass} \]

\[ a_c > g \rightarrow T_{\text{top}} = m(a_c - g) > 0 \rightarrow \text{tension is taut makes it to the top} \]

That is, in order for a mass to a taut tension and continue moving at the top of the loop, the critical value for the tension in the string \( T_{\text{top}} \) must be slightly greater than zero:

\[ T_{\text{top}} (\text{minimal}) \geq m \left( \frac{v_{\text{min}}^2}{R} - g \right) = 0 \rightarrow \quad v_{\text{min}} \geq \sqrt{gR} \quad \text{Newton’s Condition for circular motion} \]

Newton’s Condition for circular motion represents the minimum speed that an object can have at the top of the loop for it to go around the loop in circular motion. If the \( g > a_c \), the mass will decelerate too fast and not have enough “speed/centripetal acceleration” to make it to the top:

\[ T_{\text{top}} = m \left( \frac{v^2}{R} - g \right) < 0 \rightarrow \text{negative tension} \rightarrow \text{circular motion is impossible} \]
Negative tensions have no physical significance and therefore, make no sense of any type of known circular motion.

**Example 6.4**

A driver drives a truck over the top of a hill, the cross section of which can be approximated by a circle. What is the greatest speed at which he can drive without the truck leaving the road at the top of the hill?

**Solution**

At the top of the hill the vertical forces on the car are the upward normal force exerted by the ground and the downward pull of gravity. A FBD and N2L equation show that at the top,

\[ F_{c,y} = mg - N = m \frac{v^2}{R} \]

The greatest speed that the truck can have without leaving the hill is when the normal force goes to zero, which implies that the truck is not in contact with the ground anymore. Setting \( N = 0 \), sets the maximum speed:

\[ N = 0 \rightarrow mg - 0 = m \frac{v_{\text{max}}^2}{R} \rightarrow v_{\text{max}} = \sqrt{gR} = \sqrt{9.81 \cdot 250} \]

\[ = 49.5 \text{m/s} = 112 \text{ mi/hr} = v_{\text{max}} \]