1. **Parametric Surfaces:**
Recall a space curve which is defined by the vector valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$. It is a function of one variable and defines a curve in $\mathbb{R}^3$.

Now, similarly, we let our domain be a region $D$ in the $uv$-plane. Let

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k},$$

which is a vector valued function in $\mathbb{R}^3$. Still, we say $x$, $y$, and $z$ are component functions of $\mathbf{r}$, but now they are function of two variables, $u$ and $v$ with domain $D$.

The set of all points $(x, y, z)$ in $\mathbb{R}^3$ such that

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

where $u$ and $v$ are let to vary throughout $D$ is called a **parametric surface** $S$, and the above equations are the parametric equations of $S$.

2. **Example 1:** Identify the surface with vector equations $\mathbf{r}(u, v) = (v, \cos(u), 2\sin(u))$, $0 \leq v \leq 3$.

What would we get if we restricted the domain $D$ to $-\pi / 2 \leq u \leq \pi / 2$, $0 \leq v \leq 3$?
3. **Grid Curves**: A helpful tool in visualizing parametric surfaces is to consider its **grid curves**. We obtain grid curves by holding either $u$ or $v$ constant.

For example, note the grid curves from the computer generated parametric surfaces from **example 1**.

4. **Example 2 (if time, do last)**: For the parametric surface, $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, graphed below, identify which grid lines have $u$ constant and which grid lines have $v$ constant where

\[
\begin{align*}
x &= (1-u)(3+\cos v)\cos(4\pi u) \\
y &= (1-u)(3+\cos v)\sin(4\pi u) \\
z &= 3u + (1-u)\sin v
\end{align*}
\]
5. **Example 3**: For the parametric surface, \( \mathbf{r}(u,v) = (u \cos v, u \sin v, \sin u) \), graphed below, identify which grid lines have \( u \) constant and which grid lines have \( v \).

![Parametric Surface Diagram](image)

6. More often, we are given a surface, and need to find a vector function that represents that surface.

**Example 4**: Find a vector function that represents the plane that passes through the point \( P_o \) with position vector \( \mathbf{r}_o \) and that contains two nonparallel vectors \( \mathbf{a} \) and \( \mathbf{b} \).

If we write \( \mathbf{r}_o = (x_o, y_o, z_o) \), \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), then \( \mathbf{r} = (x, y, z) \) becomes:

7. **Example 5**: Find the parametric representation of the sphere \( x^2 + y^2 + z^2 = a^2 \).

![Sphere Diagram](image)
8. **Example 6**: Find the parametric representation of the surface which is the part of the cylinder \( x^2 + z^2 = 4 \) that lies between the planes \( y = -2 \) and \( y = 6 \).

9. **Example 7**: Find a parameterization of the part of the cone \( z = 2\sqrt{x^2 + y^2} \) that lies below the sphere \( x^2 + y^2 + z^2 = 10 \).

10. In general, if a surface is given as a function of \( x \) and \( y \), that is \( z = f(x, y) \), then we can parameterize the surface \( S \) by:

11. **Surface of Revolution** (if time): Consider the 5B surface obtained by rotating the curve \( y = f(x), \ a \leq x \leq b \), about the \( x \)-axis, where \( f(x) \geq 0 \). Let \( \theta \) be the angle of rotation. If \( (x, y, z) \) is a point on \( S \), then we can parametric the surface as
12. **Tangent Planes:**

Let $S$ be a surface described by the vector function

$$
\mathbf{r}(u,v) = x(u,v) \mathbf{i} + y(u,v) \mathbf{j} + z(u,v) \mathbf{k}.
$$

At the point $P_o$, whose position vector is $\mathbf{r}_o(u_o,v_o)$:

(a) holding $u$ constant $u = u_o$, we define a grid curve $C_1$ on $S \left[ C_1 : \mathbf{r}(u_o,v) \right]$. The tangent vector to $C_1$ at $P_o$ is found by

(b) similarly, holding $v$ constant $v = v_o$, we define a grid curve $C_2$ on $S$. The tangent vector to $C_2$ at $P_o$ is found by taking the partial derivative of $\mathbf{r}$ with respect to $u$ at $u_o$.

$$
\mathbf{r}_u = \mathbf{r}_u(u_o,v_o) = \frac{\partial x}{\partial u}(u_o,v_o) \mathbf{i} + \frac{\partial y}{\partial u}(u_o,v_o) \mathbf{j} + \frac{\partial z}{\partial u}(u_o,v_o) \mathbf{k}.
$$

So long as $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$, the surface is called **smooth** (it has no “corners”). For such a smooth surface, the **tangent plane** is the plane that contains the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

13. **Example 8:** Find an equation of the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = uv$ with $u = 1$, $v = 1$. 

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Section 16.6
14. **Surface Area:**

We proceed as usual, chop and add!

(a) Divide \( D \) into many many sub-rectangles \( R_{ij} \).

(b) Choose \((u_i^*, v_j^*)\) to be the bottom left-hand corner of \( R_{ij} \), because we can.

(c) \( R_{ij} \) maps to a part of the surface \( S \), \( S_{ij} \), called a patch. The point \( P_{ij} \) with position vector \( \mathbf{r}(u_i^*, v_j^*) \) will be one of the corners of \( S_{ij} \).

(d) The tangent vectors \( \mathbf{r}_u(u_i^*, v_j^*) \) and \( \mathbf{r}_v(u_i^*, v_j^*) \) are two nice tangent vectors to \( S \) that can be used to approximate \( S_{ij} \). Find an approximation for \( a \):

\[
\text{Similarly, } \mathbf{b} = \Delta v \mathbf{r}_v(u_i^*, v_j^*)
\]

Then, the area of \( S_{ij} \) is approximately: _________________________________

And, the surface area of \( S \) is thus approximated by: __________________________

**Henceforth! Definition (6):** If a smooth parametric surface \( S \) is given by the equation

\[
\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D
\]

and is covered just once as \((u, v)\) ranges through the parametric domain \( D \), then the **surface area** of \( S \) is

\[
A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA
\]

where

\[
\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}
\]
15. **Example 9**: Find the surface area of the part of the sphere of radius $a$ to the right of the plane $y = 0$.

16. **Surface Area of the Graph of a Function**: For the special case when $S$ is a surface defined by a function of the form $z = f(x,y)$ where $(x,y)$ lies in $D$ and $f$ has continuous partial derivative, we take $x$ and $y$ as parameters as before, and parameterize $S$ as

$$x = x \quad \quad y = y \quad \quad z = f(x, y)$$

Find the surface area of $S$: 
17. **Example 10**: Find the surface area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0), (0, 1), \text{ and } (2, 1)$.

18. Note, for a good read on how we have not contradicted 5B surface areas of revolution, see page 1078.