Chapter 4: Methods of Analysis

When SCT are not applicable, it’s because the circuit is neither in series or parallel. There exist extremely powerful mathematical methods that use KVL & KCL as its basis that will make circuit solving easy! In fact, some students in the past called solving circuits "enjoyable" and they actually make you seem "smart." These methods are by far the most popular way to actually solve resistive circuits - they are called Node and Mesh analysis.

- NODE analysis uses KCL to find (node) voltages → convert currents into voltages.
- MESH (or Loop) analysis uses KVL to find (mesh) currents → convert currents into voltages.

**NODE ANALYSIS with CURRENT SOURCES ONLY**

Let me remind you of several previous concepts:
1. When we talk about voltages, we really mean electric potential differences, which reference a particular reference point (usually the ground). In circuit solving, you will choose the ground that is the most convenient for the particular circuit you are dealing with.
2. A node voltage is the potential defined at a single point. For a circuit with \( n \) nodes, there are \( n \) node voltages; however, since a ground has to be defined, there are actually \( n-1 \) node voltages to solve for.
3. Node analysis in its simplest form only has current sources present.

**General Idea**

Step 1: Determine the number of nodes \( n \), label them and pick the reference. If the ground is around, choose it! Label all the other nodes \( n-1 \) as \( v_1, v_2...v_{n-1} \).
Step 2: Pick the direction of the currents (usually from high to low potential). Be consistent!
Step 3: Apply KCL and Ohm’s law to each node and rewrite all of the currents in terms of node voltages.
Step 4: Solve the equations for \( v_1, v_2... v_{n-1} \). You should get \( n-1 \) nodes and \( n-1 \) equations to solve.

**Example:**

1. There are 3 nodes including the ground, so there are \( 3-1 = 2 \) node voltages to solve for. Label them as \( v_A \) and \( v_B \).
2. Pick current directions.
3. Apply KCL and Ohm’s law to each node to get node voltages:

KCL at Node 1: \( i_{S1} = i_1 + i_3 + i_4 \) and Ohm’s law: \( \frac{v_A - 0}{R_1} = i_4 \), \( \frac{v_A - 0}{R_3} = i_3 \), \( \frac{v_A - v_B}{R_4} = i_4 \)

Substitute back into KCL:

\[
\begin{align*}
   i_1 + i_3 + i_4 &= \frac{v_A}{R_1} + \frac{v_A}{R_3} + \frac{v_A - v_B}{R_4} = i_{S1} \\
\text{standard textbook method - not in matrix form} \\
\end{align*}
\]

Mathematical trick: \( \frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4} \) \( v_A \) - \( \frac{1}{R_4} \) \( v_B \) = \( i_{S1} \)

Carlos’ method immediately gets it into matrix form

Repeating the same process for Node 2:

KCL at Node 2: \( i_4 = i_2 + i_{S2} \) and Ohm’s law: \( \frac{v_B}{R_2} = i_2 \), \( \frac{v_A - v_B}{R_4} = i_4 \)
Substitute back into KCL: (note the way I will write it out is $i_s = i_2 + i_{S2} \rightarrow i_2 - i_4 = -i_{S2}$)

$$i_2 - i_4 = \frac{v_2 - v_1 - v_2}{R_2} \rightarrow \left(\frac{1}{R_2} + \frac{1}{R_4}\right)v_2 - \left(\frac{1}{R_4}\right)v_1 = -i_{S2}$$

Since we are given current sources $i_{S1}$ and $i_{S2}$ as well as all resistor values, we end up with two equations and two unknowns to solve $v_1$ and $v_2$ with.

4. Solve for $v_1$ and $v_2$; we can then back substitute to get all the primed currents using Ohm’s law.

$$\left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right)v_A - \left(\frac{1}{R_4}\right)v_B = i_{S1}$$

$$\left(\frac{1}{R_2} + \frac{1}{R_4}\right)v_B - \left(\frac{1}{R_4}\right)v_A = -i_{S2}$$

Let’s examine this problem again but this time note the following details. The current sources have a (+) sign for the current source means it flows into the node (supplies the node) and the (−) sign flows out of the node (draws from the node).

**NODE ANALYSIS with current sources only**

Step 1: Choose a reference, label all nodes (including the ground) and determine the number of unknown node equations.

$(\text{# of unknown nodes}) = (\text{# of node equations})$

Step 2: Apply the node equation to each unknown node voltage until the number of unknown node voltages equals the number of node equations:

$$\left(\text{Sum of conductances connected to node } v_1\right) \cdot v_1 - \left(\text{Mutual conductances}\right) \cdot (v_2, v_3, \cdots, v_{n-1}) = \pm i_{\text{source}}$$

Step 3: Solve for all node voltages using matrix techniques and answer the question/problem.

This form of Node analysis is the same as that in the book, however, my method simplifies the book methods by one additional algebraic step so to get the node equations in matrix form quicker.

**SOLVING SYSTEMS OF EQUATIONS**

When using node analysis to solve circuit problem, one sets up a system of equations that must be solved, and therefore, you need to be really good at solving these. I am aware that some of you have not had linear algebra; however, these techniques are straightforward enough that you can pick it up here. What does the calculator essentially do when you enter this into your calculator?

**Two equations, two unknowns**

Given the system of equations, we can write this in matrix form:

$$\begin{align*}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{align*}$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The solution is

$$x = \frac{c_1 b_2 - c_2 b_1}{D_2}, \quad y = \frac{a_1 c_2 - a_2 c_1}{D_2}$$

where $D_2$ is the determinant for a $2 \times 2$ matrix.
Using these matrix techniques to solve node and mesh analysis setups, let’s apply to the previous two examples we just looked at.

**Example 4.1**

Determine the node voltages \((v_a, v_b)\) and \(i_{2\Omega}\) for the circuit.

**Solution**

Before doing the calculation, which direction does the current \(i_{2\Omega}\) flow?

Let’s do the numbers:

3 nodes – 1 ref = 2 node equations for \(v_1\) and \(v_2\).

Node a: \(\left(\frac{1}{2} + \frac{1}{4}\right)v_a - \left(\frac{1}{2}\right)v_b = -3\)

Node b: \(\left(\frac{1}{2} + \frac{1}{3}\right)v_b - \left(\frac{1}{2}\right)v_a = +4\)

The current through the 2-resistor is

\[ i_{2\Omega} = \frac{1}{2}(v_b - v_a) = \frac{8}{3} \text{ A} \quad \text{or} \quad i_{2\Omega} = \frac{1}{2}(v_a - v_b) = -\frac{8}{3} \text{ A} \]

**Example 4.2**

(a) Find the node voltages. (b) Find the power absorbed by the 1-mA current source.

**Solution**

a. There should be a huge red flag being waved right in front of your eyes. Note that the units on the circuit are in \(\Omega\) and mA. The easy way to compute circuits is either with \((\Omega \& A)\) or \((k\Omega \& mA)\) but not mixed together as shown in the circuit. So I will convert into \(k\Omega \& mA\). Let’s do the numbers: 3 nodes – 0 current sources = 3 node equations and 3 unknown node voltages. Define the nodes 1, 2 and 3 and apply node analysis to get

Node 1: \(\left(\frac{1}{0.1} + \frac{1}{0.1}\right)v_1 - \left(\frac{1}{0.1}\right)v_2 - \left(\frac{1}{0.1}\right)v_3 = 4\)

Node 2: \(\left(\frac{1}{0.1} + \frac{1}{0.2} + \frac{1}{0.2}\right)v_2 - \left(\frac{1}{0.1}\right)v_1 - \left(\frac{1}{0.2}\right)v_3 = -1\)

Node 3: \(\left(\frac{1}{0.1} + \frac{1}{0.2}\right)v_3 - \left(\frac{1}{0.1}\right)v_1 - \left(\frac{1}{0.2}\right)v_2 = -4\)

The system is
\[
\begin{align*}
\frac{2}{0.1}v_1 - \frac{1}{0.1}v_2 - \frac{1}{0.1}v_3 & = 4 \\
-\frac{1}{0.1}v_1 + \frac{4}{0.2}v_2 - \frac{1}{0.2}v_3 & = -1 \\
-\frac{1}{0.1}v_1 - \frac{1}{0.2}v_2 + \frac{3}{0.2}v_3 & = -4
\end{align*}
\]

\[
\begin{pmatrix}
20 & -10 & -10 \\
-10 & 20 & -5 \\
-10 & -5 & 15
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 
\begin{pmatrix}
4 \\
-1 \\
-4
\end{pmatrix}
\]

\[
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 
\begin{pmatrix}
-0.1 V \\
-0.2 V \\
-0.4 V
\end{pmatrix}
\]

b. The power of the 1mA- current sources is
\[
P_{\text{1mA}} = v_i i = (-0.2 \text{ V} \cdot 1 \text{ mA}) = -0.2 \text{ mW} = P_{\text{1mA}}
\]

Once again, it is required that you be able to solve system of equations with your calculator. If you do not know, I am sure that many students in this course already know how to do this or look online. Your ability to pass the first exam and quizzes will depend on this skill. Get it! Note that mathematics is a tool in this course; it is not our main direction. The bottom line is solving circuits and understanding them.

**NODE ANALYSIS WITH VOLTAGE SOURCES**

When using Node analysis to calculate node voltages, there are two different types of voltage sources in circuits that one must understand how to maneuver around:

- Voltage sources connected to the reference ground
- Voltage sources NOT connected to the reference ground.

Voltage sources in a circuit requires one to modify how node analysis is applied. Consider the following circuit and let’s analyze all of the node voltages without any calculations.

![Circuit Diagram](image)

As usual, we ask how many nodes are in the circuit, and we find that there are 3 nodes to solve for (not including the ground). Note that the two voltage sources in the circuit are different in how they are connected to the ground: the (1) 10V-source is connected to the ground whereas the (2) 8V-source is NOT connected to the ground.

1. **10V-source is connected to the ground**
   The left top node must be 10 V since the **10V-source is directly connected to the ground and immediately determines the value of the top left node**. Therefore, the 10V-voltage source literally reduced the number of unknown node voltages by one. So the role of a voltage source in node analysis reduces the number of node voltages by the number of voltage sources.

   \[
   \text{(Number of node equations)} = \text{(\# of Nodes)} - \text{(\# of voltage sources)}
   \]

   In this case, we have

   \[
   3 \text{ unknown nodes} - 1 \text{ source} = 2 \text{ unknown node voltages}
   \]

2. **8V-source is NOT connected to the ground**
   A voltage source not connected to the ground sets up a special situation that requires care in setting up the node equations. Whenever a voltage source is NOT connected to the ground it forms a **Supernode** voltage; that is, a supernode is formed when two (or more) nodes are connected to each via a voltage source NOT connected to the ground. **Identify**
the Super node by drawing a dashed circle around the two connected nodes.
The supernode forces node voltages \( v_a \) and \( v_b \) to be dependent upon each other and do not act independently anymore. This can be seen by writing out the voltage across the 8V-voltage source in terms of nodes voltages:

\[
v_a - v_b = 8 \quad \rightarrow \quad v_a = v_b + 8
\]

If I know the value of \( v_b \), I will automatically know the value of \( v_a \). Having the 8V-source in the circuit then further reduces the number of nodes we need to solve for:

3 nodes - 2 voltage sources = 1 unknown node voltage to solve for.

That is, although we started with 3 nodes, we now have 3 - 2 = 1 unknown node equations to solve. However, since nodes \( v_a \) and \( v_b \) are dependent on each other, we write a Supernode Condition \( \equiv \) SC. The SC tells us how these two nodes are connected to each other via the voltage source:

\[
SC: \quad v_a - v_b = 8
\]

Overall summary of Supernodes
1. The effect of a voltage source in a circuit is to reduce the number of unknown node voltages required to calculate.

\[
(Number \ of \ node \ equations) = (\ # \ of \ nodes) - (\ # \ of \ voltage \ sources)
\]

2. If a voltage source is NOT connected to the ground, then that voltage source forms a SUPERNODE and one has to modify the node equations. A Supernode is a node equation where several node equations are combined into one single supernode equation. A Supernode is typically circled with dashed lines to identify it.

\[
SN \ equation: \quad (node \ equation \ for \ node \ a) + (node \ equation \ for \ node \ b) = \pm i_{\text{source}}
\]

3. Along with every Supernode equation, there is a SUPERNODE CONDITION \( (\equiv \) SC) that explicitly shows how these two nodes are dependent on each other via the voltage source not connected to the ground:

\[
SC: \quad v_x - v_y = v_S
\]

4. Tricky scenario: if after the counting there are no node equations to be had (that is, the number of nodes is equal to the number of voltage sources in the circuit), then it is the collection of SC equations that still exist and are used to solve for the node voltages.

Node Analysis with Current and Voltage Sources
Step 1: Choose a reference and label all nodes including the “ground,” and determine the number of unknown nodes equations:

\[
(Number \ of \ node \ equations) = (\ # \ of \ nodes) - (\ # \ of \ voltage \ sources)
\]

Step 2: Apply the node equation to each unknown node voltage until the number of unknown node voltages equals the number of node equations:

\[
\left( \text{Sum of conductances connected to node } v_1 \right) \cdot v_1 - \left( \text{Mutual conductances} \right) \cdot (v_2, v_3, \cdots, v_{n-1}) = \pm i_{\text{source}}
\]

Step 3: If a voltage source is NOT connected to the ground, then that voltage source forms a SUPERNODE (and identify it by drawing a dashed loop). The two nodes that form a supernode must be written as a Supernode (SN) equation plus a Supernode Condition (SC):

\[
SN \ equation: \quad (node \ equation \ for \ node \ a) + (node \ equation \ for \ node \ b) = \pm i_{\text{source}}
\]

\[
SC: \quad v_x - v_y = v_S
\]

Step 4: Solve for all node voltages and appropriate quantities.
If we apply this process to this circuit, let’s go through these steps. Write down the Supernode equation for this circuit:

\[
\left(\frac{1}{100} + \frac{1}{100}\right) \cdot v_a - \left(\frac{1}{100}\right) \cdot 8 + \left(\frac{1}{100}\right) \cdot v_b = 0
\]

Apply the Supernode condition:

\[v_a = v_b + 8\]

Writing this into matrix form, we get

\[
\begin{bmatrix}
2 & 1 & 0 \\
0 & 100 & 100
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b
\end{bmatrix} =
\begin{bmatrix}
8 \\
100
\end{bmatrix}
\]

\[v_a = 6V, \quad v_b = -2V\]

Now we can solve for the currents in the circuit using Ohm’s law. We get

\[i = \frac{v - v_a}{100} = \frac{10 - 6}{100} = 40mA = i_a; \quad i_a = \frac{6}{100} = 60mA = \frac{v_a}{100} = \frac{v_a}{100}; \quad i_b = \frac{v_b}{100} = \frac{-2}{100} = -20mA = i_b\]

**Example 4.3**

Determine the node voltage \(v\) and \(i\) for the circuit.

**Solution**

After choosing the bottom node to be ground, there are 3 unknown nodes with one voltage source not connected to the ground, so that the nodes associated with this voltage source forms a supernode. Due to the supernode, there is automatically an associated SC. Let’s do the numbers:

3 nodes − 1 source = 2 node equations →

\[\text{SN (v}_B, v_C)\]

\[\text{regular node v}_A, \text{SC (v}_B, v_C)\]

3 eqs & 3 unknowns

\[(v_A, v_B, v_C)\]

Applying our rules for node analysis,

**Node A:** \(\left(\frac{1}{2} + \frac{1}{5}\right) \cdot v_A - \left(\frac{1}{2}\right) \cdot v_B = \frac{-1}{2}\)

**SN (v}_B, v_C):**

\[\left(\frac{1}{2} + \frac{1}{5}\right) \cdot v_B - \left(\frac{1}{2}\right) \cdot v_A - \left(\frac{1}{5}\right) \cdot v_C\]

\[= \frac{1}{2} + 1 + 2 = \frac{7}{2}\]

**SC (v}_B, v_C):** \(v_C - v_B = 2\)

The system of equations is

\[
\begin{align*}
\frac{3}{2}v_A - \frac{1}{2}v_B + 0 \cdot v_C &= -\frac{1}{2} \\
-\frac{1}{2}v_A + \frac{1}{2}v_B + \frac{1}{4}v_C &= \frac{7}{2} \\
0 \cdot v_A - 1 \cdot v_B + 1 \cdot v_C &= 2
\end{align*}
\]

\[
\begin{bmatrix}
\frac{3}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\
0 & -1 & +1
\end{bmatrix}
\begin{bmatrix}
v_A \\
v_B \\
v_C
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{2} \\
\frac{7}{2} \\
2
\end{bmatrix}
\]

\[v = v_A = 1.29V\]
For the current i, we don't need to solve any equation, and immediately apply simple circuit techniques to get it. The 2V-source is parallel to the 5Ω resistor, and therefore, has the same voltage drop. Applying Ohm's law:

\[ v = Ri \quad \Rightarrow \quad i = \frac{2}{5} = 0.4\text{A} \]

**MESH or LOOP ANALYSIS**

Node Analysis equations are generated by KCL. In Mesh Analysis the equations are generated instead by KVL to create **loop currents**. The easiest situation to start with in Mesh Analysis is having all voltage sources and later including current source into our equations.

**MESH ANALYSIS with voltage sources only**

The general idea is:

Step 1: Identify all independent loops and assign a loop current to all loops in the same direction (either CW or CCW). Here I choose CW directions.

Step 2: From the current directions, these define the resistor polarities and label them.

Step 3: Apply KVL in the CW direction to each loop and rewrite all voltages into currents using Ohm's law.

Step 4: Solve the resulting equations for each loop currents. There should be 1 equation for each independent loop.

**Example:**

KVL to loop a: \(-2 + v_{350} + 4 + v_{100} = 0\); Ohm's law: \(v_{350} = 350i_a, \quad v_{100} = 100(i_a - i_b)\)

Substitute back into KVL:

\[ -2 + 350i_a + 4 + 100(i_a - i_b) = 0 \quad \text{Mathematical trick} \rightarrow (350 + 100)i_a - (100)i_b = 2 - 4 \]

Carlos' method immediately gets it into matrix form immediately

Repeating this process again to loop b with Carlos’ trick,

\( (400 + 100)i_b - (100)i_b = 4 - 8 \)

Summarizing these system of equations, we get

\[
\begin{bmatrix}
(350 + 100)i_a - (100)i_b = 2 - 4 \\
(400 + 100)i_b - (100)i_a = 4 - 8
\end{bmatrix} \quad \text{matrix form} \rightarrow \begin{bmatrix}
450 & -100 \\
-100 & 500
\end{bmatrix} \begin{bmatrix}
i_a \\
i_b
\end{bmatrix} = \begin{bmatrix}
-2 \\
-4
\end{bmatrix} \quad \text{solution} \rightarrow \begin{bmatrix}
i_a \\
i_b
\end{bmatrix} = \begin{bmatrix}
-6.5 \text{mA} \\
-9.3 \text{mA}
\end{bmatrix}
\]

Let's step back and analyze this result:

- Sum of the resistors for the current flowing in loop-1
- If another loop is mutual, subtract those resistor values multiplied by the appropriate current.
- Place voltage source values on the other side with the understanding that

**MESH or LOOP ANALYSIS with voltage sources only**

Step 1: For each loop assign a current loop and define the polarities. Determine the number of unknown loop currents:
(# of loop equations) = (# of unknown loop currents)

Step 2: Apply the loop equation to each unknown loop current until the number of unknown loop currents equals the number of loop equations:

\[
\left( \text{Sum of resistors connected with loop-1} \right) \cdot i_i = \left( \text{Mutual resistors} \right) \cdot (i_2, i_3, \ldots, i_n) = \pm V_{sources}
\]

Step 3: Solve for all loop currents and appropriate quantities.

Example 4.4
Determine the mesh currents \((i_1, i_2, \text{ and } i_3)\) and the voltage across 3Ω-resistor for the circuit shown.

![Circuit Diagram]

Solution
There are 3 unknown loop currents (which I draw in the CW directions) and we want to solve for \(V_{3Ω}\). Now that we are more sophisticated, we should start adding a solver equation for \(V_{3Ω}\). Since we do not know the directions of the loop currents, we guess that the voltage \(V_{3Ω}\) is positive when

\[
v_{3Ω} = 3(i_1 - i_2) \quad \text{or} \quad 3i_1 - 3i_2 - V_{3Ω} = 0
\]

I now add this to my calculations for the mesh circuit. Let’s do the numbers:

3 unknown loops = 3 loop currents → loops \((i_1, i_2, i_3)\)

4 eqs & 4 unknowns

\[
\begin{align*}
(2 + 3 + 9)i_1 - (3)i_2 - (9)i_3 &= 0 \\
(3 + 6)i_2 - (3)i_1 - (6)i_3 &= -15 \\
(6 + 9)i_3 - (9)i_1 - (6)i_2 &= 21 \\
3i_1 - 3i_2 - V_3 &= 0
\end{align*}
\]

Writing these out, gives

\[
\begin{bmatrix}
14 & -3 & -9 & 0 \\
-3 & 9 & -6 & 0 \\
-9 & -6 & 15 & 0 \\
3 & -3 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
V_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
-15 \\
-21 \\
0
\end{bmatrix}
\]

solutions →

\[
i_1 = 3 \text{ A} \\
i_2 = 2 \text{ A} \\
i_3 = 4 \text{ A} \\
V_3 = 3 \text{ V}
\]

MESH ANALYSIS with voltage and current sources

When using mesh analysis to calculate loop currents, there are two flavors of current sources in circuits that one must accommodate for:

- current sources that are isolated from other loops
- current sources CONNECTING two loops together.

Current sources in a circuit requires one to modify how mesh analysis is applied.

Consider the following circuit and let’s determine all of the loop currents.

![Circuit Diagram]
As usual, we ask how many loops are in the circuit, and we find that there are 3 loops to solve for. Note that the two current sources in the circuit are different in how they are connected in the circuit: the (1) 10mA-source is isolated from the other loops whereas the (2) 20mA-source is connected between two loops and NOT isolated.

1. 10mA-source is isolated from the rest of the loops
   The left most loop current must be 10mA since the 10mA-source is isolated from the rest of the loops and immediately determines the value of the left most loop. Therefore, the 10mA-current source literally reduced the number of unknown loop currents by one. So the role of a current source in mesh analysis reduces the number of loop currents by the number of current sources.

   \[
   \text{(Number of loop equations)} = ( \# \text{ of loops}) - ( \# \text{ of current sources})
   \]
   In this case, we have
   
   3 unknown loops – 1 source = 2 unknown loop currents

2. 20mA-source IS connected to other loops
   A current source connected to other loops sets up a special situation that requires care in setting up the mesh equations. Whenever a current source is CONNECTS two loops together it forms a Superloop (or Supermesh) current; that is, a superloop is formed when two loops are connected to each via a current source. Identify the Superloop by drawing a dashed circle around the two connected loops.

   The superloop forces loop currents \( i_2 \) and \( i_3 \) to be dependent upon each other now and do not act independently anymore. This can be seen by writing out the current through the 20mA-current source:

   \[
   i_3 - i_2 = 20 \quad \rightarrow \quad i_3 = i_2 + 20
   \]

   If I know the value of \( i_2 \), I will automatically know the value of \( i_3 \). Having the 20mA-source in the circuit then further reduces the number of loops we need to solve for:

   \[
   \text{(Number of mesh equations)} = ( \# \text{ of loops}) - ( \# \text{ of current sources})
   \]

   3 loops – 2 current sources = 1 unknown loop current to solve for

   That is, although we started with 3 loops, we now have \( 3 - 2 = 1 \) unknown loop equation to solve. However, since loops \( i_2 \) and \( i_3 \) are dependent on each other, we write a Superloop Condition \( \equiv \text{SC} \). The SC tells us how these two loops are connected to each other via the current source:

   \[
   \text{SC:} \quad i_3 - i_2 = 20
   \]

   If I know the value of \( v_b \), I will automatically know the value of \( v_a \). Having the 8V-source in the circuit then further reduces the number of nodes we need to solve for:

   \[
   \text{3 nodes – 2 voltage sources} = 1 \text{ unknown node voltage to solve for}
   \]

   That is, although we started with 3 nodes, we now have \( 3 - 2 = 1 \) unknown node equations to solve. However, since nodes \( v_a \) and \( v_b \) are dependent on each other, we write a Supernode Condition \( \equiv \text{SC} \). The SC tells us how these two nodes are connected to each other via the voltage source:

   \[
   \text{SC:} \quad v_a - v_b = 8
   \]

   Analogy: Superloops are analogous to Supernodes in Node Analysis. When a voltage source was included in the circuit and Node Analysis is done, the number of node voltages that one needed to solve for was reduced by the number of voltage sources in the circuit. A similar situation exists for Mesh Analysis.
**Overall summary of Supernodes**

5. The effect of a voltage source in a circuit is to reduce the number of unknown node voltages required to calculate.

\[(\text{Number of node equations}) = (\# \text{ of nodes}) - (\# \text{ of voltage sources})\]

6. If a voltage source is NOT connected to the ground, then that voltage source forms a **SUPERNODE** and one has to modify the node equations. A Supernode is a node equation where several node equations are combined into one single supernode equation. A **Supernode is typically circled with dashed lines to identify it.**

<table>
<thead>
<tr>
<th>SN equation:</th>
<th>(node equation for node a) + (node equation for node b)</th>
<th>= ±I_{source}</th>
</tr>
</thead>
<tbody>
<tr>
<td>supernode equation</td>
<td></td>
<td></td>
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</tbody>
</table>

7. Along with every Supernode equation, there is a **SUPERNODE CONDITION** (≡ SC) that explicitly shows how these two nodes are dependent on each other via the voltage source not connected to the ground:

\[\text{SC: } v_x - v_y = V_S\]

8. Tricky scenario: if after the counting there are no node equations to be had (that is, the number of nodes is equal to the number of voltage sources in the circuit), then it is the collection of SC equations that still exist and are used to solve for the node voltages.

**Overall summary of Superloops**

1. The effect of a current source in a circuit is to reduce the number of unknown loop currents required to calculate.

\[(\text{Number of mesh equations}) = (\# \text{ of loops}) - (\# \text{ of current sources})\]

2. If a current source is NOT isolated from other loops, then that current source forms a **SUPERLOOP** and one has to modify the mesh equations. A superloop is a mesh equation where several mesh equations are combined into one single superloop equation. A Superloop is typically circled with dashed lines to identify it.

<table>
<thead>
<tr>
<th>SL equation:</th>
<th>(mesh equation for loop-1) + (mesh equation for loop-2)</th>
<th>= ±V_{source}</th>
</tr>
</thead>
<tbody>
<tr>
<td>superloop equation</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Along with every Superloop equation, there is a **SUPERLOOP CONDITION** (≡ SC) that explicitly shows how these two loops are dependent on each other via the current source connecting the two loops together:

\[\text{SC: } i_x - i_y = I_S\]

4. Tricky scenario: if after the counting there are no loop equations to be had (that is, the number of loops is equal to the number of current sources in the circuit), then it is the series of SC equations that still exist and are used to solve for the currents.

**Mesh (or Loop) Analysis with Current and Voltage Sources**

Step 1: For each loop assign a current loop and define the polarities. Determine the number of unknown loop currents:

\[(\text{Number of mesh equations}) = (\# \text{ of loops}) - (\# \text{ of current sources})\]

Step 2: Apply the loop equation to each unknown loop current until the number of unknown loop currents equals the number of loop equations:

\[\left(\text{Sum of resistors connected with loop-1}\right)i_1 - \left(\text{Mutual resistors}\right)(i_2, i_3, \ldots, i_n) = ±V_{sources}\]

Step 3: If a current source is NOT isolated from other loops, then that current source forms a **SUPERLOOP (and identify it by drawing a dashed loop)** and the two loops from Step 2 must be written as a Superloop (SL) equation plus a Superloop Condition (SC):
SL equation: $(\text{mesh equation for loop-1}) + (\text{mesh equation for loop-2}) = \pm v_{\text{source}}$

Superloop equation

$SC: \quad i_x - i_y = i_s$

Step 4: Solve for all loop currents and appropriate quantities.

**Example 4.5** Find the current $i_1$, $v_x$ and $v_z$ using Mesh Analysis.

**Solution**

There are 3 loops in this circuit and there is one current source connecting two loops that forms a superloop. Due to the superloop, this automatically implies that there is also a SC associated with the superloop. Let's do the numbers:

3 loops $(i_1, i_2, i_3) - 1$ current source $= 2$ mesh equations $\rightarrow \begin{cases} \text{regular loop } i_x \\ SL(i_1 & i_3) \\ SC(i_1 & i_3) \end{cases}$

So there are 2 loop equations (regular plus a superloop) and 1 SC.

Writing these out, we get

Superloop: $(1+1+1)i_3 - (1)i_2 = 10$

SC: $i_1 - i_3 = 1$

Loop-2: $(1+1+10)i_2 - (1)i_1 - (1)i_3 = 0$

Solver: $v_x = i_3 - i_2$

\[
\begin{pmatrix}
0 & -1 & 3 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 7 & -4 & 0
\end{pmatrix}
\begin{pmatrix}
i_1 \\
i_2 \\
i_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
10 \\
0
\end{pmatrix}

Solution:

\[
i_1 = 4.44 \text{ mA}
\]

\[
i_2 = 0.656 \text{ mA}
\]

\[
i_3 = 3.44 \text{ mA}
\]

\[
v_x = 2.78 \text{ V}
\]

**NODE & MESH ANALYSIS with DEPENDENT SOURCES**

When there are circuits with dependent sources, we treat the dependent source exactly as any other source when performing node or mesh analysis. However, an additional equation must be generated via the dependent source condition (≡ DC) in order to solve the circuit. Here is the process:

1. Apply regular Node or Mesh analysis
2. If a circuit contains a dependent source, apply the DC:
   - Go to the controlling element and see how it is defined
   - The controlling current or voltage of the dependent source MUST be written as a function of the
   - Node voltages in node analysis
   - Loop currents in mesh analysis
   - As before, usually it DC comes in the form of Ohm's law but also expect KCL and KVL.

**Node Analysis with Dependent Sources**

Determine $v_0$ using Node analysis.
There are 2 nodes but there is one voltage. However, we have a dependent current source that automatically implies there is also a DC. Let’s do the numbers:

\[
\begin{align*}
\text{2 nodes - 1 source = 1 node equation} & \rightarrow \begin{cases} 
\text{regular node} \ v_0 \\
\text{DC} \ (i_x) 
\end{cases} \\
2 \text{ eqs and 2 unknowns} \ (v_0 & \ v_x)
\end{align*}
\]

Apply the regular node analysis steps (& treating the dependent source as a regular source) and write out

\[
\text{Node } v_0 : \left( \frac{1}{4} + \frac{1}{2} \right) v_0 - \left( \frac{1}{4} \right) \cdot 2 = 5i_x \longrightarrow \left( \frac{1}{4} + \frac{1}{2} \right) v_0 - 5i_x = \frac{i_x}{2}
\]

The way it stands this is an unsolved equation, finding another equation remedies the problem. The dependent source condition gives us that second equation to make it possible to solve for \(v_0\).

To apply the DC, we look at how the controlling element \(i_x\) is defined; it’s the current through the 4kΩ resistor. This implies that we rewrite \(i_x\) in terms of the node voltages for node analysis – \(v_0\). Using Ohm’s law, we rewrite \(i_x\) in terms of \(v_0\):

\[
2 - v_0 = 4i_x
\]

Writing down our two equations gives

\[
\begin{align*}
\left( \frac{1}{4} + \frac{1}{2} \right) v_0 - 5i_x = \frac{1}{2} \\
v_0 + 4i_x = 2 
\end{align*}
\]

\[
\begin{pmatrix} \frac{1}{4} + \frac{1}{2} & -5 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} v_0 \\ i_x \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} v_0 \\ i_x \end{pmatrix} = \begin{pmatrix} 1.5V \\ \frac{1}{2} \end{pmatrix}
\]

Example 4.6
Determine the voltage \(v_x\) for this node circuit.

Solution
Where should I place the ground? There are 3 loops in this circuit and there is one current source connecting two loops that forms a superloop. Due to the superloop, this automatically implies that there is also a SC associated with the superloop.

Let’s do the numbers:

\[
\begin{align*}
\text{4 nodes - 3 sources = 1 node equation} & \rightarrow \begin{cases} 
\text{SN} \ (v_1, \ v_2) \\
\text{SC} \ (v_1, \ v_2) \\
\text{DC} \ (i) \\
\text{Solver} \ (v_x)
\end{cases} \\
4 \text{ eqs and 4 unknowns} \ (v_1, \ v_2, \ i, \ v_x)
\end{align*}
\]
Note that this is a supernode since it is not connected to the ground. Speaking of the ground, which ground do I choose? The most convenient node is usually the one where most of the voltage sources are connected. I would choose the node where the 8V and 4V node is located.

Step 1: Apply regular node analysis treating the dependent source as a regular source

Supernode: \( \frac{1}{2}v_1 - \frac{1}{2} \cdot 8 + (\frac{1}{2})v_2 = 4 \)

Supernode condition: \( v_2 = 6i + v_1 \)

Step 2: Dependent Source Condition

The controlling element \( i \) is defined as the current through the \( 2\Omega \) resistor. This implies that we rewrite this controlling current in terms of node voltages for node analysis. Using Ohm’s law, we have

Dependent Source Condition: \( v_2 - 0 = 2i \)

Putting this altogether, gives

\[
\begin{align*}
\frac{1}{2}v_1 + \frac{1}{2}v_2 + 0 &= 8 \\
- v_1 + v_2 - 6i &= 0 \\
0 + \frac{1}{2}v_2 - i &= 0
\end{align*}
\]

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
-1 & 1 & -6 \\
0 & \frac{1}{2} & -1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
i
\end{pmatrix}
= \begin{pmatrix}
8 \\
0 \\
0
\end{pmatrix}
\]

Solution

\[
v_1 = 32V \quad v = v_1 - 8 = 24V
\]

MESH ANALYSIS WITH DEPENDENT SOURCES

Analyze the simple circuit using Mesh analysis – determine the loop current \( i_a \) for the circuit.

Let’s do the numbers:

(2 loops) – (1 source) = 1 loop equation + DC → 2 equations + 2 unknowns

- **Apply regular mesh analysis treating the dependent source as a regular source.** There is only one independent loop.

  \[
  \text{Loop-}i_0: \quad (200 + 100)i_0 - (100) \cdot 4i = -8 \quad \rightarrow \quad (300)i_0 - (400) \cdot i_x = -8
  \]

- **Dependent Source Condition:** The controlling element \( i_0 \) is defined as the current through the \( 100\Omega \) resistor. Since we are not interested in voltages but loop currents, this implies that we rewrite this controlling current in terms of loop currents for mesh analysis. Using KCL, we have

  DC: \( 4i_x = i_x + i_0 \rightarrow -i_0 + 3i_x = 0 \)

Putting this altogether gives

\[
\begin{pmatrix}
300 & -400 & -8 \\
-1 & 3 & 0
\end{pmatrix}
\begin{pmatrix}
i_0 \\
i_x
\end{pmatrix}
= \begin{pmatrix}
-8 \\
0
\end{pmatrix}
\]

Solution

\[
i_0 = -48mA
\]

**Example 4.7**

Determine the loop current \( i_x \) and the voltage \( v_c \) across the CCCS for the mesh circuit.

Solution
Label the loops starting from the left as loop 1, upper one 2, lower one 3, and the right one $i_x$ going in the CCW direction. Note that voltage $v_c$ is the voltage drop across the VCCS and $i_x$ is one of the loop currents.

Let’s do the numbers:

4 loops – 2 sources = 2 loop equations →

- regular loop ($i_2$)
- SL ($i_1$, $i_2$, $i_x$)
- 2 SC: ($i_1$, $i_y$) & ($i_3$, $i_x$)
- DC ($i_x$, $v_y$)
- Solver ($v_c$)
- 6 eqs and 6 unknowns ($i_1$, $i_2$, $i_3$, $i_x$, $v_y$, $v_c$)

- Apply mesh analysis – note that because the $i_x$ current is going CCW, I am going to choose the currents in those directions so that I don’t need to setup another equation.

Loop 2: $(1 + 2 + 1)i_2 - 2i_1 - i_3 = 3i_x$

Superloop: $2i_1 - 2i_2 + i_3 - i_2 + 2i_x = -3i_x - 2$

Superloop conditions

<table>
<thead>
<tr>
<th>loop 1 &amp; 3:</th>
<th>loop 1</th>
<th>loop 3</th>
<th>loop $i_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_3 = i_1 + 9$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i_3 = 2v_y + i_x$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Dependent Source Condition – there are two of them. The CCVS is written in terms of the loop current $i_x$. This one is already met when we chose the loop currents to move in the CCW direction. The second one is the VCCS which depends on $v_y$ – the voltage across the left 2Ω resistor. In terms of loop currents, $v_y$ is

Dependent Source Condition: $v_y = 2(i_2 - i_1)$

- Solver equation: applying KVL to get $v_c$, we get

$$v_c = -2i_x - 3i_x$$

Putting this altogether gives

$$\begin{bmatrix} -2 & -2 & -1 & -3 & 0 & 0 \\ 2 & -3 & 1 & 5 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_x \\ v_y \\ v_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

solutions

$$\begin{align*}
i_x &= -1A \\
v_c &= 5V
\end{align*}$$