Null or Empty Set

Some sets do not contain any elements, such as the set of zebras that live in your house.

Definition: Empty Set
The set that contains no elements is called the empty set or null set and is symbolized by \( \{ \} \) or \( \emptyset \).

Note that \( \{\emptyset\} \) is not the empty set. This set contains the element \( \emptyset \) and has a cardinality of 1. The set \( \{0\} \) is also not the empty set because it contains the element 0. It also has a cardinality of 1.

Example 8 Natural Number Solutions
Indicate the set of natural numbers that satisfies the equation \( x + 2 = 0 \).

Solution The values that satisfy the equation are those natural numbers that make the equation a true statement. Only the number \(-2\) satisfies this equation. Because \(-2\) is not a natural number, the solution set of this equation is \( \{ \} \) or \( \emptyset \).

Universal Set

Another important set is a universal set.

Definition: Universal Set
A universal set, symbolized by \( U \), is a set that contains all the elements for any specific discussion.

When a universal set is given, only the elements in the universal set may be considered when working the problem. If, for example, the universal set for a particular problem is defined as \( U = \{1, 2, 3, 4, \ldots, 10\} \), then only the natural numbers 1 through 10 may be used in that problem.

SECTION 2.1 Exercises

Warm Up Exercises

In Exercises 1–12, fill in the blank with an appropriate word, phrase, or symbol(s).

1. A collection of objects is called a(n) ________.

2. Three dots placed in a set to show that the set continues in the same manner is called a(n) ________.

3. The three ways a set can be written are ________, ________, and ________.

4. A set that contains no elements or the number of elements in the set is a natural number is called a(n) ________ set.

5. A set that is not finite is called a(n) ________ set.

6. Two sets that contain the same elements are called ________ sets.

7. Two sets that contain the same number of elements are called ________ sets.

8. The number of elements in a set is called the ________ number.

9. The set that contains no elements is called the ________ set.

10. The two ways to indicate an empty set are ________ and ________.

11. A set that contains all the elements for any specific discussion is called a(n) ________ set.

12. Two sets that have the same cardinal number can be placed in a(n) ________ correspondence.
Practice the Skills

In Exercises 13–18, determine whether each set is well defined or not well defined.

13. The set of the best colleges
14. The set of the most interesting courses at your school
15. The set of states that have a common border with Kansas
16. The set of the four states in the United States having the largest population on January 1, 2010
17. The set of astronauts who walked on the moon

\[ E = \{ x \mid x \in N \text{ and } 14 \leq x < 85 \} \]

34. The set of states in the United States that are not in the contiguous 48 states

In Exercises 35–38, use the following table, which shows the attendance, in millions, at the 10 most visited museums in the world in 2008. Let the 10 museums in the list represent the universal set.

<table>
<thead>
<tr>
<th>Museum</th>
<th>Attendance (in millions)</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Louvre Museum</td>
<td>8.50</td>
<td>Paris, France</td>
</tr>
<tr>
<td>2. British Museum</td>
<td>5.93</td>
<td>London, UK</td>
</tr>
<tr>
<td>3. National Gallery of Art</td>
<td>4.96</td>
<td>Washington, DC</td>
</tr>
<tr>
<td>4. Tate Modern</td>
<td>4.95</td>
<td>London, UK</td>
</tr>
<tr>
<td>5. Metropolitan Museum of Art</td>
<td>4.82</td>
<td>New York, NY</td>
</tr>
<tr>
<td>6. Vatican Museums</td>
<td>4.44</td>
<td>Vatican City</td>
</tr>
<tr>
<td>7. National Gallery</td>
<td>4.38</td>
<td>London, UK</td>
</tr>
<tr>
<td>8. Musee d’Orsay</td>
<td>3.03</td>
<td>Paris, France</td>
</tr>
<tr>
<td>9. Musee d’Art Moderne Prado</td>
<td>2.98</td>
<td>Paris, France</td>
</tr>
<tr>
<td>10. Museum of Modern Art</td>
<td>2.90</td>
<td>New York, NY</td>
</tr>
</tbody>
</table>

Source: The Art Newspaper

Use the list to determine each set in roster form.

35. The set of museums in which the attendance was more than 4.5 million
36. The set of museums in which the attendance was less than 3 million
37. The set of museums in which the attendance was between 2 million and 4 million
38. The set of museums in which the attendance was between 3.5 million and 5.5 million

In Exercises 39–42, use the graph on page 49, which shows iPod sales, in millions, for the years 2003–2008.

Use the graph to determine each set in roster form.

39. The set of years in which iPod sales were more than 52 million
40. The set of years in which iPod sales were less than 8 million
41. The set of years in which iPod sales were between 8 million and 60 million
42. The set of years in which iPod sales were more than 65 million

<table>
<thead>
<tr>
<th>Year</th>
<th>iPod Sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>1.5</td>
</tr>
<tr>
<td>2004</td>
<td>8.3</td>
</tr>
<tr>
<td>2005</td>
<td>32.0</td>
</tr>
<tr>
<td>2006</td>
<td>46.4</td>
</tr>
<tr>
<td>2007</td>
<td>52.7</td>
</tr>
<tr>
<td>2008</td>
<td>63.2</td>
</tr>
</tbody>
</table>

In Exercises 43–50, express each set in set-builder notation.

43. \( B = \{7, 8, 9, 10, 11, 12, 13, 14\} \)
44. \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \)
45. \( C = \{3, 6, 9, 12, \ldots\} \)
46. \( D = \{5, 10, 15, 20, \ldots\} \)
47. \( E \) is the set of odd natural numbers.
48. \( A \) is the set of national holidays in the United States in July.
49. \( C \) is the set of months that contain less than 30 days.
50. \( F = \{15, 16, 17, \ldots, 100\} \)

In Exercises 51–58, write a description of each set.

51. \( A = \{1, 2, 3, 4, 5, 6, 7\} \)
52. \( D = \{3, 6, 9, 12, 15, 18, \ldots\} \)
53. \( V = \{a, e, i, o, u\} \)
54. \( S = \{\text{Bashful, Doe, Dopey, Grumpy, Happy, Sleepy, Sneezy}\} \)

55. \( T = \{\text{oak, maple, elm, pine, \ldots}\} \)
56. \( E = \{x \mid x \in N \text{ and } 4 \leq x < 11\} \)
57. \( S = \{\text{spring, summer, fall, winter}\} \)
58. \( B = \{\text{John Lennon, Ringo Starr, Paul McCartney, George Harrison}\} \)

In Exercises 59–62, use the following list, which shows the 10 countries with the most cellular subscribers, in millions, as of 2008. Let the 10 countries in the list represent the universal set.

<table>
<thead>
<tr>
<th>Country</th>
<th>Number of Subscribers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. China</td>
<td>649.70</td>
</tr>
<tr>
<td>2. India</td>
<td>376.12</td>
</tr>
<tr>
<td>3. United States</td>
<td>260.00</td>
</tr>
<tr>
<td>4. Russia</td>
<td>172.00</td>
</tr>
<tr>
<td>5. Brazil</td>
<td>151.90</td>
</tr>
<tr>
<td>6. Indonesia</td>
<td>115.60</td>
</tr>
<tr>
<td>7. Japan</td>
<td>102.98</td>
</tr>
<tr>
<td>8. Germany</td>
<td>101.50</td>
</tr>
<tr>
<td>9. Pakistan</td>
<td>91.40</td>
</tr>
<tr>
<td>10. United Kingdom</td>
<td>70.00</td>
</tr>
</tbody>
</table>

Source: CIA

Use the list to determine each set in roster form.

59. \( \{x \mid x \text{ is a country with at least 250 million cellular subscribers}\} \)
60. \( \{x \mid x \text{ is a country with fewer than 100 million cellular subscribers}\} \)
61. \( \{x \mid x \text{ is a country with between 100 million and 200 million cellular subscribers}\} \)
62. \( \{x \mid x \text{ is a country with between 250 million and 500 million cellular subscribers}\} \)
In Exercises 63–66, use the following graph, which shows the cost of a 30-second commercial during the Super Bowl from 1996 to 2011. Let the 16 years represent the universal set.

Use the graph to represent each set in roster form.

63. The set of years in which the cost of a Super Bowl commercial was more than $2.5 million
64. The set of years in which the cost of a Super Bowl commercial was less than $2.0 million
65. The set of years in which the cost of a Super Bowl commercial was between $2.0 million and $2.5 million
66. The set of years in which the cost of a Super Bowl commercial was more than $2.5 million and less than $3 million

Cost of a 30-Second Commercial During the Super Bowl

Source: www.industry.bnet.com

In Exercises 67–74, state whether each statement is true or false. If false, give the reason.

67. \( \{ e \} \subseteq \{ a, e, i, o, u \} \)
68. \( b \in \{ a, b, c, d, e, f \} \)
69. \( h \in \{ a, b, c, d, e, f \} \)
70. Mickey Mouse \( \in \{ \text{characters created by Walt Disney} \} \)
71. \( 3 \in \{ x \mid x \in N \text{ and } x \text{ is odd} \} \)
72. Amazon \( \in \{ \text{rivers in the United States} \} \)
73. Titanic \( \in \{ \text{top 10 motion pictures with the greatest revenues} \} \)
74. \( 2 \in \{ x \mid x \text{ is an odd natural number} \} \)

In Exercises 75–78, for the sets \( A = \{ 2, 4, 6, 8 \} \), \( B = \{ 1, 3, 7, 9, 13, 21 \} \), \( C = \{ \#, \& \} \), and \( D = \{ \# , \& \% \} \), determine \( n(A) \), \( n(B) \), \( n(C) \), and \( n(D) \).

77. Determine \( n(C) \).
78. Determine \( n(D) \).

In Exercises 79–84, determine whether the pairs of sets are equal, equivalent, both, or neither.

79. \( A = \{ \text{algebra, geometry, trigonometry} \} \), \( B = \{ \text{geometry, trigonometry, algebra} \} \)
80. \( A = \{ 7, 9, 10 \} \), \( B = \{ a, b, c \} \)
81. \( A = \{ \text{grapes, apples, oranges} \} \), \( B = \{ \text{grapes, peaches, apples, oranges} \} \)
82. \( A \) is the set of Siamese cats.
\( B \) is the set of cats.
83. \( A \) is the set of letters in the word *bank*.
\( B \) is the set of letters in the word *post*.
84. \( A \) is the set of states.
\( B \) is the set of state capitals.

Problem Solving

85. Set-builder notation is often more versatile and efficient than listing a set in roster form. This versatility is illustrated with the following two sets.

\( A = \{ x \mid x \in N \text{ and } x > 2 \} \)
\( B = \{ x \mid x > 2 \} \)

a) Write a description of set \( A \) and set \( B \).

b) Explain the difference between set \( A \) and set \( B \).
\( \text{(Hint: Is } 4 \frac{1}{2} \in A? \text{ Is } 4 \frac{1}{2} \in B? \text{)} \)

c) Write set \( A \) in roster form.

d) Can set \( B \) be written in roster form? Explain your answer.

86. Consider sets \( A \) and \( B \) below

\( A = \{ x \mid 2 < x \leq 5 \text{ and } x \in N \} \)
\( B = \{ x \mid 2 < x \leq 5 \} \)

a) Write a description of set \( A \) and set \( B \).

b) Explain the difference between set \( A \) and set \( B \).

c) Write set \( A \) in roster form.

d) Can set \( B \) be written in roster form? Explain your answer.

A cardinal number answers the question "How many?" An ordinal number describes the relative position that an element occupies. For example, Molly's desk is the third desk from the aisle.
In Exercises 87–90, determine whether the number used is a cardinal number or an ordinal number.

87. J. K. Rowling has written 7 Harry Potter books.

88. Study the chart on page 25 in the book.

89. Lincoln was the sixteenth president of the United States.

90. Emily paid $35 for her new blouse.

91. Describe three sets of which you are a member.

92. Describe three sets that have no members.

93. Write a short paragraph explaining why the universal set and the empty set are necessary in the study of sets.

Challenge Problem/Group Activity

94. a) In a given exercise, a universal set is not specified, but we know that actor Orlando Bloom is a member of the universal set. Describe five different possible universal sets of which Orlando Bloom is a member.

b) Write a description of one set that includes all the universal sets in part (a).

Internet/Research Activity

95. Georg Cantor is recognized as the founder and a leader in the development of set theory. Do research and write a paper on his life and his contributions to set theory and to the field of mathematics. References include history of mathematics books, encyclopedias, and the Internet.

SECTION 2.2 Subsets

Consider the following sets. Set A = {baseball, basketball, hockey}. Set B = {baseball, football, basketball, hockey, softball}. Note that each element of set A is also an element of set B. In this section, we will discuss how to illustrate the relationship between two sets, A and B, when each element of set A is also an element of set B.

**Why This is Important** The relationship between sets is important throughout life. For example, to gain a promotion at work, you may need to fulfill different sets of criteria.

In our complex world, we often break larger sets into smaller, more manageable sets, called subsets. For example, consider the set of people in your class. Suppose we categorize the set of people in your class according to the first letter of their last name (the A’s, B’s, C’s, etc.). When we do so, each of these sets may be considered a subset of the original set. Each of these subsets can be separated further. For example, the set of people whose last name begins with the letter A can be categorized as either male or female or by their age. Each of these collections of people is also a subset. A given set may have many different subsets.

**Definition: Subset**

Set A is a subset of set B, symbolized by \( A \subseteq B \), if and only if all the elements of set A are also elements of set B.

The symbol \( A \subseteq B \) indicates that “set A is a subset of set B.” The symbol \( \nsubseteq \) is used to indicate “is not a subset.” Thus, \( A \nsubseteq B \) indicates that set A is not a subset of set B. To show that set A is not a subset of set B, we must find at least one element of set A that is not an element of set B.
2.2 Subsets 55

SECTION 2.2 Exercises

Warm Up Exercises

In Exercises 1–4, fill in the blank with an appropriate word, phrase, or symbol(s).

1. If all the elements of set $A$ are also elements of set $B$, then $A$ is a(n) ______ of set $B$.

2. If all the elements of set $A$ are also elements of set $B$, and $A \neq \emptyset$, then set $A$ is a(n) ______ of set $B$.

3. The expression for determining the number of distinct subsets for a set with $n$ distinct elements is _______.

4. The expression for determining the number of distinct proper subsets for a set with $n$ distinct elements is _______.

Practice the Skills

In Exercises 5–26, answer true or false. If false, give the reason.

5. $\{\text{book}\} \subseteq \{\text{magazine, newspaper, book}\}$

6. $\{\text{Italy}\} \subseteq \{\text{Italy, Spain, France, Switzerland, Austria}\}$

7. $\{\text{McIntosh, Red Delicious\} \subseteq \{\text{Empire, Gala, Cortland, Red Delicious\}}$

8. $\{\text{pepper, salt\} \subseteq \{\text{salt, butter, mayonnaise\}}$

9. $\{\text{motorboat, kayak\} \subset \{\text{kayak, fishing boat, motorboat, sailboat\}}$

10. $\{\text{polar bear, tiger, lion\} \subset \{\text{tiger, lion, polar bear, penguin\}}$

11. $\{4, 2, 7\} \subset \{4, 7, 2\}$

12. $\{c, a, r, t\} \subset \{t, r, a, c\}$

13. $\text{Xbox 360} \in \{\text{PSIII, Wii, Xbox 360\}}$

14. $\text{LaGuardia} \in \{\text{JFK, LaGuardia, Newark\}}$

15. $\{\text{swimming\} \in \{\text{sailing, water skiing, swimming\}}$

16. $\{\\} \in \{1, 3, 5, 7\}$

17. $5 \notin \{2, 4, 6\}$

18. $\{\\} \subseteq \{\text{table, chair, sofa\}}$

19. $\{\text{red\} \subset \{\text{red, blue, green\}}$

20. $\{3, 5, 9\} \subset \{3, 9, 5\}$

21. $\{\\} = \{\emptyset\}$

22. $\emptyset = \{\}$

23.

24. $\{\\} \subseteq \{\}$

25. $0 = \{\}$

26. $\{1\} \subseteq \{\{1\}, \{2\}, \{3\}\}$

In Exercises 27–34, determine whether $A = B$, $A \subseteq B$, $B \subseteq A$, $A \subset B$, $B \subset A$, or if none of these applies. (There may be more than one answer.)

27. $A = \{\text{penny, nickel, dime, quarter\} B = \{\text{penny, quarter\}}$

28. $A = \{x \mid x \in N \text{ and } x < 6\}$
   $B = \{x \mid x \in N \text{ and } 1 \leq x \leq 5\}$

29. Set $A$ is the set of states that border the Atlantic Ocean. Set $B$ is the set of states east of the Mississippi River.

30. $A = \{1, 3, 5, 7, 9\}$
   $B = \{3, 9, 5, 7, 6\}$

31. $A = \{x \mid x \text{ is a brand of soft drink\} B = \{\text{A & W, Coca-Cola, Dr Pepper, Mountain Dew\}}$

32. $A = \{x \mid x \text{ is a sport that uses a ball\} B = \{\text{basketball, soccer, tennis\}}$

33. Set $A$ is the set of natural numbers between 2 and 7. Set $B$ is the set of natural numbers greater than 2 and less than 7.

34. Set $A$ is the set of all cars manufactured by General Motors. Set $B$ is the set of sports cars manufactured by General Motors.

In Exercises 35–38, list all the subsets of the sets given.

35. $D = \emptyset$

36. $A = \{\emptyset\}$

37. $B = \{\text{cow, horse\}}$
38. \( C = \{\text{steak, pork, chicken}\} \)

Problem Solving

39. For set \( A = \{a, b, c, d\} \),
   
   a) list all the subsets of set \( A \).
   
   b) state which of the subsets in part (a) are not proper subsets of set \( A \).

40. A set contains nine elements.
   
   a) How many subsets does it have?
   
   b) How many proper subsets does it have?

In Exercises 41--52, if the statement is true for all sets \( A \) and \( B \), write “true.” If it is not true for all sets \( A \) and \( B \), write “false.” Assume that \( A \neq \emptyset \), \( U \neq \emptyset \), and \( A \subseteq U \).

41. If \( A \subseteq B \), then \( A \subseteq C \).
42. If \( A \subseteq B \), then \( A \subseteq C \).
43. \( A \subseteq A \)
44. \( A \subseteq A \)
45. \( \emptyset \subseteq A \)
46. \( \emptyset \subseteq A \)
47. \( A \subseteq U \)
48. \( \emptyset \subseteq \emptyset \)
49. \( \emptyset \subseteq U \)
50. \( U \subseteq \emptyset \)
51. \( \emptyset \subseteq \emptyset \)
52. \( U \subseteq \emptyset \)

53. Ordering a Pizza  Jasmine Sullivan is ordering a pizza at Domino’s Pizza. She can add any of the following toppings: olives, pepperoni, sausage, onions, green peppers, mushrooms, anchovies, and ham. How many different variations of the pizza and toppings can be made?

54. Building a House  The Jacobsens are planning to build a house in a new development. They can either build the base model offered by the builder or add any of the following options: deck, hot tub, security system, hardwood flooring. How many different variations of the house are possible?

55. Salad Toppings  Donald Wheeler is ordering a salad at a Ruby Tuesday restaurant. He can purchase a salad consisting of just lettuce, or he can add any of the following items: cucumber, onion, tomato, carrot, green pepper, olive, mushroom. How many different variations of a salad are possible?

56. Telephone Features  A customer with Verizon can order telephone service with some, all, or none of the following features: call waiting, call forwarding, caller identification, three-way calling, voice mail, fax line. How many different variations of the set of features are possible?

57. If \( E \subseteq F \) and \( F \subseteq E \), what other relationship exists between \( E \) and \( F \)?

58. How can you determine whether the set of boys is equivalent to the set of girls at a roller-skating rink?

59. For the set \( D = \{a, b, c\} \)
   
   a) is \( a \) an element of set \( D \)?
   
   b) is \( c \) a subset of set \( D \)?
   
   c) is \( \{a, b\} \) a subset of set \( D \)?

Challenge Problem/Group Activity

60. Hospital Expansion  A hospital has four members on the board of directors: Arnold, Benitez, Cathy, and Dominique.
   
   a) When the members vote on whether to add a wing to the hospital, how many different ways can they vote (abstentions are not allowed)? For example, Arnold—yes, Benitez—no, Cathy—no, and Dominique—yes is one of the many possibilities.
   
   b) Make a listing of all the possible outcomes of the vote. For example, the vote described in part (a) could be represented as (YNNY).
   
   c) How many of the outcomes given in part (b) would result in a majority supporting the addition of a wing to the hospital? That is, how many of the outcomes have three or more Y’s?

Recreational Mathematics

61. How many elements must a set have if the number of proper subsets of the set is \( \frac{1}{2} \) of the total number of subsets of the set?

62. If \( A \subseteq B \) and \( B \subseteq C \), must \( A \subseteq C \)?

63. If \( A \subseteq B \) and \( B \subseteq C \), must \( A \subseteq C \)?

64. If \( A \subseteq B \) and \( B \subseteq C \), must \( A \subseteq C \)?

Internet/Research Activity

65. On page 53, we discussed the ladder of life. Do research and indicate all the different classifications in the Linnæan system, from most general to the most specific, in which a koala belongs.
d) To determine \( A - C' \), we must first determine \( C' \).

\[
C' = \{ a, c, d, f, h, i, j, k \}
\]

\( A - C' \) is the set of elements that are in set \( A \) but not set \( C' \). The elements that are in set \( A \) but not set \( C' \) are \( b, e, \) and \( g \). Therefore, \( A - C' = \{ b, e, g \} \).

Next we discuss the Cartesian product.

**Cartesian Product**

Definition: **Cartesian Product**

The **Cartesian product** of set \( A \) and set \( B \), symbolized by \( A \times B \) and read “\( A \) cross \( B \),” is the set of all possible ordered pairs of the form \((a, b)\), where \( a \in A \) and \( b \in B \).

To determine the ordered pairs in a Cartesian product, select the first element of set \( A \) and form an ordered pair with each element of set \( B \). Then select the second element of set \( A \) and form an ordered pair with each element of set \( B \). Continue in this manner until you have used each element of set \( A \).

**Example 10** The **Cartesian Product of Two Sets**

Given \( A = \{ \text{orange, banana, apple} \} \) and \( B = \{ 1, 2 \} \), determine the following.

a) \( A \times B \)

b) \( B \times A \)

c) \( A \times A \)

d) \( B \times B \)

**Solution**

a) \( A \times B = \{ (\text{orange}, 1), (\text{orange}, 2), (\text{banana}, 1), (\text{banana}, 2), (\text{apple}, 1), (\text{apple}, 2) \} \)

b) \( B \times A = \{ (1, \text{orange}), (1, \text{banana}), (1, \text{apple}), (2, \text{orange}), (2, \text{banana}), (2, \text{apple}) \} \)

c) \( A \times A = \{ (\text{orange}, \text{orange}), (\text{orange}, \text{banana}), (\text{orange}, \text{apple}), (\text{banana}, \text{orange}), (\text{banana}, \text{banana}), (\text{banana}, \text{apple}), (\text{apple}, \text{orange}), (\text{apple}, \text{banana}), (\text{apple}, \text{apple}) \} \)

d) \( B \times B = \{ (1, 1), (1, 2), (2, 1), (2, 2) \} \)

We can see from Example 10 that, in general, \( A \times B \neq B \times A \). The ordered pairs in \( A \times B \) are not the same as the ordered pairs in \( B \times A \) because \((\text{orange}, 1) \neq (1, \text{orange})\).

In general, if a set \( A \) has \( m \) elements and a set \( B \) has \( n \) elements, then the number of ordered pairs in \( A \times B \) will be \( m \times n \). In Example 10, set \( A \) contains 3 elements and set \( B \) contains 2 elements. Notice that \( A \times B \) contains \( 3 \times 2 \) or 6 ordered pairs.

### Warm Up Exercises

In exercises 1–8, fill in the blank with an appropriate word, phrase, or symbol(s).

1. The set of all the elements in the universal set that are not in set \( A \) is called the _____ of set \( A \).

2. The set containing all the elements that are members of set \( A \) or of set \( B \) or of both sets is called the _____ of set \( A \) and set \( B \).

3. The set containing all the elements that are common to both set \( A \) and set \( B \) is called the _____ of set \( A \) and set \( B \).

4. The set of elements that belong to set \( A \), but not to set \( B \), is called the _____ of two sets \( A \) and \( B \).

5. The set of all possible ordered pairs of the form \((a, b)\), where \( a \in A \) and \( b \in B \), is called the _____ product of set \( A \) and set \( B \).
6. If set $A$ has $m$ elements and set $B$ has $n$ elements, the Cartesian product $A \times B$ has _______ elements.

7. Two sets with no elements in common are called _______ sets.

8. In a Venn diagram with two overlapping sets there are _______ regions.

Practice the Skills

In Exercises 9–13, use Fig. 2.2 as a guide to draw a Venn diagram that illustrates the situation described.

9. Set $A$ and set $B$ are disjoint sets.

10. $A \subset B$

11. $B \subset A$

12. $A = B$

13. Set $A$ and set $B$ are overlapping sets.

14. Which set operation is the word or generally interpreted to mean?

15. Which set operation is the word and generally interpreted to mean?

16. Give the relationship between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$.

Problem Solving

17. **Cellular Telephones** For the sets $U$, $A$, and $B$, construct a Venn diagram and place the elements in the proper regions.

   $U = \{ \text{iPhone, Blackberry, LG, DROID, Samsung, Nokia, Motorola, Sony} \}$

   $A = \{ \text{iPhone, Blackberry, LG, Motorola, DROID} \}$

   $B = \{ \text{LG, DROID, Nokia, Motorola} \}$

18. **National Parks** For the sets $U$, $A$, and $B$, construct a Venn diagram and place the elements in the proper regions.

   $U = \{ \text{Badlands, Death Valley, Glacier, Grand Teton, Mammoth Cave, Mount Rainier, North Cascades, Shenandoah, Yellowstone, Yosemite} \}$

   $A = \{ \text{Badlands, Glacier, Grand Teton, Mount Rainier, Yellowstone} \}$

   $B = \{ \text{Death Valley, Glacier, Mammoth Cave, Mount Rainier, Yosemite} \}$

19. **Occupations** The following table shows the fastest-growing occupations for college graduates, based on employment in 2008 and the estimated employment in 2016. Let the occupations in the table represent the universal set.

<table>
<thead>
<tr>
<th>Occupation</th>
<th>Employment (in thousands of jobs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biomedical engineers (BE)</td>
<td>16</td>
</tr>
<tr>
<td>Network systems analysts (NSA)</td>
<td>292</td>
</tr>
<tr>
<td>Financial examiners (FE)</td>
<td>27</td>
</tr>
<tr>
<td>Medical scientists (MS)</td>
<td>109</td>
</tr>
<tr>
<td>Physicians assistants (PA)</td>
<td>75</td>
</tr>
<tr>
<td>Biochemists (B)</td>
<td>23</td>
</tr>
<tr>
<td>Athletic trainers (AT)</td>
<td>16</td>
</tr>
<tr>
<td>Dental hygienists (DH)</td>
<td>174</td>
</tr>
<tr>
<td>Veterinary technicians (VT)</td>
<td>80</td>
</tr>
<tr>
<td>Computer software engineers (CSE)</td>
<td>515</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Occupation</th>
<th>2016</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biomedical engineers (BE)</td>
<td>28</td>
</tr>
<tr>
<td>Network systems analysts (NSA)</td>
<td>448</td>
</tr>
<tr>
<td>Financial examiners (FE)</td>
<td>38</td>
</tr>
<tr>
<td>Medical scientists (MS)</td>
<td>154</td>
</tr>
<tr>
<td>Physicians assistants (PA)</td>
<td>104</td>
</tr>
<tr>
<td>Biochemists (B)</td>
<td>32</td>
</tr>
<tr>
<td>Athletic trainers (AT)</td>
<td>22</td>
</tr>
<tr>
<td>Dental hygienists (DH)</td>
<td>237</td>
</tr>
<tr>
<td>Veterinary technicians (VT)</td>
<td>108</td>
</tr>
<tr>
<td>Computer software engineers (CSE)</td>
<td>690</td>
</tr>
</tbody>
</table>

   Source: U.S. Bureau of Labor Statistics

Let $A =$ the set of fastest-growing occupations for college graduates whose 2008 employment was at least 80,000.

Let $B =$ the set of fastest-growing occupations for college graduates whose estimated employment in 2016 is at least 200,000.

Using the abbreviations listed in the table for each occupation, construct a Venn diagram illustrating the sets.
20. \textbf{Racing Standings} The following table shows the 2009 NASCAR Sprint Cup Series Final Standings, with the 10 drivers having the highest point total and the number of races won. Let the drivers in the table represent the universal set.

<table>
<thead>
<tr>
<th>Driver</th>
<th>Points</th>
<th>Wins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jimmie Johnson</td>
<td>6652</td>
<td>7</td>
</tr>
<tr>
<td>Mark Martin</td>
<td>6511</td>
<td>5</td>
</tr>
<tr>
<td>Jeff Gordon</td>
<td>6473</td>
<td>1</td>
</tr>
<tr>
<td>Kurt Busch</td>
<td>6446</td>
<td>2</td>
</tr>
<tr>
<td>Denny Hamlin</td>
<td>6335</td>
<td>4</td>
</tr>
<tr>
<td>Tony Stewart</td>
<td>6309</td>
<td>4</td>
</tr>
<tr>
<td>Greg Biffle</td>
<td>6292</td>
<td>0</td>
</tr>
<tr>
<td>Juan Montoya</td>
<td>6252</td>
<td>0</td>
</tr>
<tr>
<td>Ryan Newman</td>
<td>6175</td>
<td>0</td>
</tr>
<tr>
<td>Kasey Kahne</td>
<td>6128</td>
<td>2</td>
</tr>
</tbody>
</table>

Source: NASCAR

Let $A$ = the set of drivers with more than 6400 points

Let $B$ = the set of drivers with more than 1 win

Construct a Venn diagram illustrating the sets. Use the driver's initials in the Venn diagram.

21. Let $U$ represent the set of animals in U.S. zoos. Let $A$ represent the set of animals in the San Diego zoo. Describe $A'$.

\begin{center}
\textbf{San Diego Zoo}
\end{center}

22. Let $U$ represent the set of U.S. colleges and universities. Let $A$ represent the set of U.S. colleges and universities in the state of Mississippi. Describe $A'$.

In Exercises 23–28,

$U$ is the set of farms in the United States.

$A$ is the set of farms that produce corn.

$B$ is the set of farms that produce tomatoes.

Describe each of the following sets in words.

23. $A'$

24. $B'$

25. $A \cup B$

26. $A \cap B$

27. $A \cap B'$

28. $A \cup B'$

In Exercises 29–34,

$U$ is the set of furniture stores.

$A$ is the set of furniture stores that sell mattresses.

$B$ is the set of furniture stores that sell outdoor furniture.

$C$ is the set of furniture stores that sell leather furniture.

Describe the following sets.

29. $A \cup C$

30. $A \cap B$

31. $B' \cap C$

32. $A \cap B \cap C$

33. $A \cup B \cup C$

34. $A' \cup C'$

In Exercises 35–42, use the Venn diagram in Fig. 2.12 to list the set of elements in roster form.

\begin{center}
\textbf{Figure 2.12}
\end{center}

35. $A$

36. $B$

37. $A \cap B$

38. $U$
39. $A \cup B$

40. $(A \cup B)'$

41. $A' \cap B'$

42. $(A \cap B)'$

In Exercises 43–50, use the Venn diagram in Fig. 2.13 to list the set of elements in roster form.

**Figure 2.13**

43. $A$

44. $B$

45. $U$

46. $A \cap B$

47. $A' \cup B$

48. $A \cup B'$

49. $A' \cap B$

50. $(A \cup B)'$

In Exercises 51–60, let

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 2, 4, 5, 7\}$

$B = \{2, 3, 5, 6\}$

Determine the following.

51. $A \cup B$

52. $A \cap B$

53. $B'$

54. $A \cup B'$

55. $(A \cup B)'$

56. $A' \cap B'$

57. $(A \cup B) \cap B$

58. $(A \cup B) \cap (A \cup B)'$

59. $(B \cup A)' \cap (B' \cup A')$

60. $A' \cup (A \cap B)$

In Exercises 61–70, let

$U = \{a, b, c, d, e, f, g, h, i, j, k\}$

$A = \{a, c, d, f, g, i\}$

$B = \{b, c, d, f, g\}$

$C = \{a, b, f, i, j\}$

Determine the following.

61. $B'$

62. $B \cup C$

63. $A \cap C$

64. $A' \cup B'$

65. $(A \cap C)'$

66. $(A \cap B) \cup C$

67. $A \cup (C \cap B)'$

68. $A \cup (C' \cup B')$

69. $(A' \cup C) \cup (A \cap B)$

70. $(C \cap B) \cap (A' \cap B)$

In Exercises 71–78, let

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 4, 6, 9\}$

$B = \{1, 3, 4, 5, 8\}$

$C = \{4, 5, 9\}$

Determine the following.

71. $A - B$

72. $A - C$

73. $A - B'$

74. $A' - C$

75. $(A - B)'$

76. $(A - B') - C$

77. $C - A'$

78. $(C - A') - B$

In Exercises 79–84, let

$A = \{a, b, c\}$

$B = \{1, 2\}$

79. Determine $A \times B$.

80. Determine $B \times A$.

81. Does $A \times B = B \times A$?

82. Determine $n(A \times B)$.

83. Determine $n(B \times A)$.

84. Does $n(A \times B) = n(B \times A)$?

**Problem Solving**

In Exercises 85–98, let

$U = \{x | x \in N \text{ and } x < 10\}$

$A = \{x | x \in N \text{ and } x \text{ is odd and } x < 10\}$

$B = \{x | x \in N \text{ and } x \text{ is even and } x < 10\}$

$C = \{x | x \in N \text{ and } x < 6\}$

Determine the following.

85. $A \cap B$

86. $A \cup B$

87. $A' \cup B$

88. $(B \cup C)'$

89. $A \cap C'$

90. $A \cap B'$

91. $(B \cap C)'$

92. $(A \cup C) \cap B$

93. $(A \cap C) \cap B$

94. $(C \cap B) \cup A$

95. $(A \cap B) \cup C$

96. $(A' \cup C) \cap B$

97. $(A' \cup B') \cap C$

98. $(A' \cap C) \cup (A \cap B)$

99. When will a set and its complement be disjoint? Explain and give an example.
100. When will \( n(A \cap B) = 0 \)? Explain and give an example.

101. Pet Ownership The results of a survey of customers at PetSmart showed that 27 owned dogs, 38 owned cats, and 16 owned both dogs and cats. How many people owned either a dog or a cat?

102. Student Council and Intramurals At Madison High School, 46 students participated in student council or intramurals, 30 participated in student council, and 4 participated in student council and intramurals. How many students participated in intramurals?

103. Consider the formula
\[
n(A \cup B) = n(A) + n(B) - n(A \cap B)
\]

a) Show that this relation holds for \( A = \{a, b, c, d\} \) and \( B = \{b, d, e, f, g, h\} \).

b) Make up your own sets \( A \) and \( B \), each consisting of at least six elements. Using these sets, show that the relation holds.

c) Use a Venn diagram and explain why the relation holds for any two sets \( A \) and \( B \).

104. The Venn diagram in Fig. 2.14 shows a technique of labeling the regions to indicate membership of elements in a particular region. Define each of the four regions with a set statement. (Hint: \( A \cap B' \) defines region I.)

\[
\begin{align*}
U & \quad A & \quad B \\
I & \quad x \in A \quad x \in B' \\
II & \quad x \in A \quad x \in B \\
III & \quad x \in A' \quad x \in B' \\
IV & \quad x \in A' \quad x \in B
\end{align*}
\]

In Exercises 105–114, let \( U = \{0, 1, 2, 3, 4, 5, \ldots\} \), \( A = \{1, 2, 3, 4, \ldots\} \), \( B = \{4, 8, 12, 16, \ldots\} \), and \( C = \{2, 4, 6, 8, \ldots\} \). Determine the following.

105. \( A \cup B \)

106. \( A \cap B \)

107. \( B \cup C \)

108. \( B \cap C \)

109. \( A \cap C \)

110. \( A' \cap C \)

111. \( B' \cap C \)

112. \( (B \cup C)' \cup C \)

113. \( (A \cap C) \cap B' \)

114. \( U' \cap (A \cup B) \)

Challenge Problems/Group Activities

In Exercises 115–122, determine whether the answer is \( \emptyset \), \( A \), or \( U \). (Assume \( A \neq \emptyset, A \neq U \).)

115. \( A \cap A' \)

116. \( A \cup A' \)

117. \( A \cup \emptyset \)

118. \( A \cap \emptyset \)

119. \( A' \cup U \)

120. \( A \cap U \)

121. \( A \cup U \)

122. \( A \cap A \)

In Exercises 123–128, determine the relationship between set \( A \) and set \( B \) if

123. \( A \cap B = B \).

124. \( A \cup B = B \).

125. \( A \cap B = \emptyset \).

126. \( A \cup B = A \).

127. \( A \cap B = A \).

128. \( A \cup B = \emptyset \).
A financial planning company uses the Venn diagram above to illustrate the financial planning services it offers. From the diagram, we can see that this company offers advice in an “intersection” of the areas investment, retirement, and college planning, the intersection of all three sets.

We categorize items on a daily basis. Children are taught how to categorize items at an early age when they learn how to classify items according to color, shape, and size. Biologists categorize items when they classify organisms according to shared characteristics.

Why This is important A Venn diagram is a very useful tool to help order and arrange items and to picture the relationship between sets.

<table>
<thead>
<tr>
<th>Set</th>
<th>Corresponding Regions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>I, II, IV, V</td>
</tr>
<tr>
<td>B ∩ C</td>
<td>V, VI</td>
</tr>
<tr>
<td>A ∪ (B ∩ C)</td>
<td>I, II, IV, V, VI</td>
</tr>
</tbody>
</table>

Find \( A \cup (B \cap C) \)

<table>
<thead>
<tr>
<th>Set</th>
<th>Corresponding Regions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>I, II, III, IV, V, VI</td>
</tr>
<tr>
<td>A ∪ C</td>
<td>I, II, IV, V, VI, VII</td>
</tr>
<tr>
<td>(A ∪ B) ∩ (A ∪ C)</td>
<td>I, II, IV, V, VI</td>
</tr>
</tbody>
</table>

Find \( (A \cup B) \cap (A \cup C) \)

The regions that correspond to \( A \cup (B \cap C) \) are I, II, IV, V, and VI, and the regions that correspond to \( (A \cup B) \cap (A \cup C) \) are also I, II, IV, V, and VI. The results show that both statements are represented by the same regions, namely, I, II, IV, V, and VI, and therefore \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \) for all sets \( A, B, \) and \( C \).

In Example 4, we proved that \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \) for all sets \( A, B, \) and \( C \). Show that this statement is true for the specific sets \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( A = \{1, 2, 3, 7\} \), \( B = \{2, 3, 4, 5, 7, 9\} \), and \( C = \{1, 4, 7, 8, 10\} \).

De Morgan’s Laws

In set theory, logic, and other branches of mathematics, a pair of related theorems known as De Morgan’s laws make it possible to transform statements and formulas into alternative and often more convenient forms. In set theory, De Morgan’s laws are symbolized as follows.

De Morgan’s Laws

1. \( (A \cap B)' = A' \cup B' \)
2. \( (A \cup B)' = A' \cap B' \)

Law 2 was verified in Example 3. We suggest that you verify law 1 at this time. The laws were expressed verbally by William of Ockham in the fourteenth century. In the nineteenth century, Augustus De Morgan expressed them mathematically. De Morgan’s laws will be discussed more thoroughly in Chapter 3, Logic.

SECTION 2.4

Exercises

Warm Up Exercises

In Exercises 1–4, fill in the blank with an appropriate word, phrase, or symbol(s).

1. The number of regions created when constructing a Venn diagram with three overlapping sets is ________.

2. a) When constructing a Venn diagram with three overlapping sets, region ________ is generally completed first.

   b) When constructing a Venn diagram with three overlapping sets, after completing region V, the next regions generally completed are II, IV, and ________.

3. Complete DeMorgan’s laws:

   a) \( (A \cup B)' = \) ________

   b) \( (A \cap B)' = \) ________

4. When using Venn diagrams to verify or determine whether set statements are equal we use ________ reasoning.

Practice the Skills/Problem Solving

5. A Venn diagram contains three sets, \( A, B, \) and \( C \), as in Fig. 2.15 on page 68. If region V contains 4 elements and there are 12 elements in \( B \cap C \), how many elements belong in region VI? Explain.
6. A Venn diagram contains three sets, $A$, $B$, and $C$, as in Fig. 2.15 on page 68. If region $V$ contains 4 elements and there are 9 elements in $A \cap B$, how many elements belong in region $I$? Explain.

7. a) For $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 4, 5\}$, and $B = \{1, 4, 5\}$, does $A \cup B = A \cap B$?

b) By observing the answer to part (a), can we conclude that $A \cup B = A \cap B$ for all sets $A$ and $B$? Explain.

c) Using a Venn diagram, determine if $A \cup B = A \cap B$ for all sets $A$ and $B$.

8. Construct a Venn diagram illustrating the following sets.
   \[ U = \{a, b, c, d, e, f, g, h, i, j\} \]
   \[ A = \{c, d, e, g, h, i\} \]
   \[ B = \{a, c, d, g\} \]
   \[ C = \{c, f, j\} \]

9. Construct a Venn diagram illustrating the following sets.
   \[ U = \{Cinderella, Pinocchio, Ratatouille, Fantasia, Dumbo, Bambi, Pocahontas, Hercules, Mulan, Tarzan, Cars\} \]
   \[ A = \{Bambi, Hercules, Pocahontas, Tarzan\} \]
   \[ B = \{Ratatouille, Bambi, Mulan, Hercules\} \]
   \[ C = \{Pocahontas, Cinderella, Bambi, Ratatouille, Fantasia\} \]

10. Construct a Venn diagram illustrating the following sets.
    \[ U = \{microwave oven, freezer, dishwasher, refrigerator, washer, dryer, toaster, blender, food processor, iron\} \]
    \[ A = \{toaster, blender, iron, dishwasher, washer, dryer\} \]
    \[ B = \{dishwasher, iron, freezer\} \]
    \[ C = \{washer, dryer, iron, freezer, microwave oven\} \]

11. Construct a Venn diagram illustrating the following sets.
    \[ U = \{American Eagle, Best Buy, Wal-Mart, Kmarts, Target, Sears, JCPenney, Costco, Kohl's, Gap, Gap Kids, Foot Locker, Old Navy, Macy's\} \]
    \[ A = \{American Eagle, Wal-Mart, Target, JCPenney, Old Navy\} \]
    \[ B = \{Best Buy, Target, Costco, Old Navy, Macy's\} \]

12. Construct a Venn diagram illustrating the following sets.
    \[ U = \{Louis Armstrong, Glenn Miller, Stan Kenton, Charlie Parker, Duke Ellington, Benny Goodman, Count Basie, John Coltrane, Dizzy Gillespie, Miles Davis, Thelonius Monk\} \]
    \[ A = \{Stan Kenton, Count Basie, Dizzy Gillespie, Duke Ellington, Thelonious Monk\} \]
    \[ B = \{Louis Armstrong, Glenn Miller, Count Basie, Duke Ellington, Miles Davis\} \]
    \[ C = \{Count Basie, Miles Davis, Stan Kenton, Charlie Parker, Duke Ellington\} \]

13. Olympic Medals Consider the following table, which shows countries that won at least 25 medals in the 2008 Summer Olympics. Let the countries in the table represent the universal set.

<table>
<thead>
<tr>
<th>Country</th>
<th>Gold Medals</th>
<th>Silver Medals</th>
<th>Bronze Medals</th>
<th>Total Medals</th>
</tr>
</thead>
<tbody>
<tr>
<td>United States</td>
<td>36</td>
<td>38</td>
<td>36</td>
<td>110</td>
</tr>
<tr>
<td>China</td>
<td>51</td>
<td>21</td>
<td>28</td>
<td>100</td>
</tr>
<tr>
<td>Russia</td>
<td>23</td>
<td>21</td>
<td>28</td>
<td>72</td>
</tr>
<tr>
<td>Great Britain</td>
<td>19</td>
<td>13</td>
<td>15</td>
<td>47</td>
</tr>
<tr>
<td>Australia</td>
<td>14</td>
<td>15</td>
<td>17</td>
<td>46</td>
</tr>
<tr>
<td>Germany</td>
<td>16</td>
<td>10</td>
<td>15</td>
<td>41</td>
</tr>
<tr>
<td>France</td>
<td>7</td>
<td>16</td>
<td>17</td>
<td>40</td>
</tr>
<tr>
<td>South Korea</td>
<td>13</td>
<td>8</td>
<td>10</td>
<td>31</td>
</tr>
<tr>
<td>Italy</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>28</td>
</tr>
<tr>
<td>Ukraine</td>
<td>7</td>
<td>5</td>
<td>15</td>
<td>27</td>
</tr>
<tr>
<td>Japan</td>
<td>9</td>
<td>6</td>
<td>10</td>
<td>25</td>
</tr>
</tbody>
</table>

Source: United States Olympic Committee.

Let $A =$ set of countries that won at least 30 gold medals.
Let $B =$ set of countries that won at least 15 silver medals.
Let $C =$ set of countries that won at least 10 bronze medals.

Construct a Venn diagram that illustrates the sets $A$, $B$, and $C$.

14. Popular TV Shows Construct a Venn diagram illustrating the following sets.
    \[ U = \{American Idol (AI), CSI, Dancing with the Stars (DWS), Family Guy (FG), Gossip Girl (GG), Monday Night Football (MNF), NCIS, Sunday Night Football (SNF), Survivor (S)\} \]
    \[ A = \{AI, CSI, DWS, SNF, NCIS\} \]
    \[ B = \{AI, DWS, SNF, NCIS, MNF\} \]
    \[ C = \]
Rankings of Fruit-Producing Countries  For Exercises 15–20, use the following table, which shows the top 10 countries for production of apples, oranges, and nuts. The universal set is the set of countries listed in the world.

<table>
<thead>
<tr>
<th>Apples</th>
<th>Oranges</th>
<th>Nuts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. China</td>
<td>1. Brazil</td>
<td>1. United States</td>
</tr>
<tr>
<td>2. United States</td>
<td>2. United States</td>
<td>2. Indonesia</td>
</tr>
<tr>
<td>4. Turkey</td>
<td>4. India</td>
<td>4. Ethiopia</td>
</tr>
<tr>
<td>5. Russia</td>
<td>5. China</td>
<td>5. China</td>
</tr>
</tbody>
</table>

Source: Food and Agriculture of the United Nations

Indicate in which region, I–VIII in Fig. 2.21, each of the following countries belongs.

15. Italy
16. United States
17. Canada
18. Portugal
19. Spain
20. Mexico

Figures in Exercises 21–32, indicate in Fig. 2.22 the region in which each of the figures would be placed.

Senate Bills During a session of the U.S. Senate, three bills were voted on. The votes of six senators are shown below the figure. Determine in which region of Fig. 2.23 each senator would be placed. The set labeled Bill 1 represents the set of senators who voted yes on Bill 1, and so on.

SENATOR | BILL 1 | BILL 2 | BILL 3
---|---|---|---
33. Hutchinson | yes | no | no
34. Kerry | no | no | yes
35. McCain | no | no | no
36. Mikulski | yes | yes | yes
37. Rand | no | yes | yes
38. Reid | no | yes | no

In Exercises 39–52, use the Venn diagram in Fig. 2.24 to list the sets in roster form.
In Exercises 53–60, use Venn diagrams to determine whether the following statements are equal for all sets $A$ and $B$.

53. $(A \cap B)'$, \hspace{1cm} A' \cup B'$
54. $(A \cap B)'$, \hspace{1cm} A' \cup B
55. $A' \cup B'$, \hspace{1cm} A \cap B
56. $(A \cup B)'$, \hspace{1cm} (A \cap B)'
57. $A' \cap B'$, \hspace{1cm} (A \cup B)'
58. $A' \cap B$, \hspace{1cm} A \cup B'
59. $(A' \cap B)'$, \hspace{1cm} A \cup B'
60. $A' \cap B'$, \hspace{1cm} (A' \cap B)'

In Exercises 61–70, use Venn diagrams to determine whether the following statements are equal for all sets $A$, $B$, and $C$.

61. $A \cap (B \cup C)$, \hspace{1cm} (A \cap B) \cup C
62. $A \cup (B \cap C)$, \hspace{1cm} (B \cap C) \cup A
63. $A \cap (B \cup C)$, \hspace{1cm} (B \cup C) \cap A
64. $A \cup (B \cap C)'$, \hspace{1cm} A' \cap (B' \cup C)
65. $A \cap (B \cup C)$, \hspace{1cm} (A \cap B) \cup (A \cap C)
66. $A \cup (B \cap C)$, \hspace{1cm} (A \cup B) \cap (A \cup C)
67. $A \cup (B \cap C)'$, \hspace{1cm} A \cup (B' \cap C')
68. $(A \cup B) \cap (B \cup C)$, \hspace{1cm} B \cup (A \cap C)
69. $(A \cup B)' \cap C$, \hspace{1cm} (A' \cup C') \cap (B' \cup C)
70. $(C \cap B)' \cup (A \cap B)'$, \hspace{1cm} A \cap (B \cup C)

In Exercises 71–74, use set statements to write a description of the shaded area. Use union, intersection, and complement as necessary. More than one answer may be possible.

71. \hspace{1cm} 72. \hspace{1cm} 73. \hspace{1cm} 74.

75. Let

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{1, 2, 3, 4\}$
$B = \{3, 6, 7\}$
$C = \{6, 7, 9\}$

a) Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for these sets.

b) Make up your own sets $A$, $B$, and $C$. Verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for your sets $A$, $B$, and $C$.

c) Use Venn diagrams to verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for all sets $A$, $B$, and $C$.

76. Let

$U = \{a, b, c, d, e, f, g, h, i\}$
$A = \{a, c, d, e, f\}$
$B = \{c, d\}$
$C = \{a, b, c, d, e\}$

a) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for these sets.

b) Make up your own sets $A$, $B$, and $C$. Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for your sets.

c) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for all sets $A$, $B$, and $C$.

77. Blood Types

A hematology text gives the following information on percentages of the different types of blood worldwide.

<table>
<thead>
<tr>
<th>Type</th>
<th>Positive Blood, %</th>
<th>Negative Blood, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>37</td>
<td>6</td>
</tr>
<tr>
<td>O</td>
<td>32</td>
<td>6.5</td>
</tr>
<tr>
<td>B</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>AB</td>
<td>5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Construct a Venn diagram similar to the one in Example 2 and place the correct percentage in each of the eight regions.

78. Define each of the eight regions in Fig. 2.25 using sets $A$, $B$, and $C$ and a set operation. (Hint: $A \cap B' \cap C'$ defines region I.)

[Diagram of Venn diagram with regions labeled I, II, III, IV, V, VI, VII, VIII]
79. Categorizing Contracts J & C Mechanical Contractors wants to classify its projects. The contractors categorize set A as construction projects, set B as plumbing projects, and set C as projects with a budget greater than $300,000.

a) Draw a Venn diagram that can be used to categorize the company projects according to the listed criteria.

b) Determine the region of the diagram that contains construction projects and plumbing projects with a budget greater than $300,000. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

c) Determine the region of the diagram that contains plumbing projects with a budget greater than $300,000 that are not construction projects. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

d) Determine the region of the diagram that contains construction projects and nonplumbing projects whose budget is less than or equal to $300,000. Describe the region using sets A, B, and C with set operations. Use union, intersection, and complement as necessary.

Challenge Problem/Group Activity
80. We were able to determine the number of elements in the union of two sets with the formula

\[ n(A \cup B) = n(A) + n(B) - n(A \cap B). \]

Can you determine a formula for finding the number of elements in the union of three sets? In other words, write a formula to determine \[ n(A \cup B \cup C). \] [Hint: The formula will contain each of the following: \[ n(A), n(B), n(C), n(A \cap B \cap C'), n(A \cap B' \cap C), n(A' \cap B \cap C), \text{ and } 2n(A \cap B \cap C). \]

Recreational Mathematics
81. a) Construct a Venn diagram illustrating four sets, A, B, C, and D. (Hint: Four circles cannot be used, and you should end up with 16 distinct regions.) Have fun!

b) Label each region with a set statement (see Exercise 78). Check all 16 regions to make sure that each is distinct.

Internet/Research Activity
82. The two Venn diagrams below illustrate what happens when colors are added or subtracted. Do research in an art text, an encyclopedia, the Internet, or another source and write a report explaining the creation of the colors in the Venn diagrams, using such terms as union of colors and subtraction (or difference) of colors.

![Venn Diagrams for Color Mixing]

SECTION 2.5 Applications of Sets

The members of a health club were surveyed about taking fitness classes at the club. Suppose the results of the survey show how many members took a yoga class, how many members took a spinning class, and how many members took a class in yoga and a class in spinning. How can the manager of the club use this information to determine how many members took only a yoga class? In this section, we will learn how to use Venn diagrams to answer this type of question.

Why This is Important As you read through this section, you will see many real-life applications of set theory.

We can solve practical problems involving sets by using the problem-solving process discussed in Chapter 1: Understand the problem, devise a plan, carry out the plan, and then examine and check the results. First determine: What is the problem? Or What am I looking for? To devise the plan, list all the facts that are given and how they are related. Look for key words or phrases such as "only set A," "set A and set B," "set A or set B," "set A and set B and not set C." Remember that and means intersection, or means union, and not means complement. The problems we solve in
SECTION 2.5  
Exercises

Practice The Skills/Problem Solving

In Exercises 1–15, draw a Venn diagram to obtain the answers.

1. Market Purchases  During the fall festival at Wambach's Farmer's market, 200 customers made the following purchases.
   - 109 purchased pumpkins.
   - 98 purchased pies.
   - 61 purchased both pumpkins and pies.

   Of those surveyed,
   a) how many purchased only pumpkins?
   b) how many purchased only pies?
   c) how many did not purchase either of these items?

2. Landscape Purchases  Agway Lawn and Garden collected the following information regarding purchases from 130 of its customers.
   - 74 purchased shrubs.
   - 70 purchased trees.
   - 41 purchased both shrubs and trees.

   Of those surveyed,
   a) how many purchased only shrubs?
   b) how many purchased only trees?
   c) how many did not purchase either of these items?

3. Real Estate  The Maellos are moving to Wilmington, Delaware. Their real estate agent located 83 houses listed for sale, in the Wilmington area, in their price range. Of these houses listed for sale,
   - 47 had a family room.
   - 42 had a deck.
   - 30 had a family room and a deck.

   How many had
   a) a family room but not a deck?
   b) a deck but not a family room?
   c) either a family room or a deck?

4. Racing  Fleet Foot Racing interviewed 150 long-distance runners to determine the type of races in which they participated. The following information was determined.
   - 102 participated in a marathon.
   - 93 participated in a triathlon.
   - 55 participated in both a marathon and a triathlon.
   
   How many
   a) participated in only a marathon?
   b) participated in only a triathlon?
   c) participated in either a marathon or a triathlon?
   d) had not participated in either a marathon or a triathlon?

5. Cultural Activities  Thirty-three U.S. cities were researched to determine whether they had a professional sports team, a symphony, or a children's museum. The following information was determined.
   - 16 had a professional sports team.
   - 17 had a symphony.
   - 15 had a children's museum.
   - 11 had a professional sports team and a symphony.
   - 7 had a professional sports team and a children's museum.
   - 9 had a symphony and children's museum.
   - 5 had all three activities.

   How many of the cities surveyed had
   a) only a professional sports team?
   b) a professional sports team and a symphony, but not a children's museum?
   c) a professional sports team or a symphony?
   d) a professional sports team or a symphony, but not a children's museum?
   e) exactly two of the activities?

6. Amusement Parks  In a survey of 85 amusement parks, it was found that
   - 24 had a hotel on site.
   - 55 had water slides.
   - 38 had a wave pool.
   - 13 had a hotel on site and water slides.
   - 10 had a hotel on site and a wave pool.
   - 19 had water slides and a wave pool.
   - 7 had all three features.
How many of the amusement parks surveyed had
a) only water slides?
b) exactly one of these features?
c) at least one of these features?
d) exactly two of these features?
e) none of these features?

7. Book Purchases A survey of 85 customers was taken at Barnes & Noble regarding the types of books purchased. The survey found that
- 44 purchased mysteries.
- 33 purchased science fiction.
- 29 purchased romance novels.
- 13 purchased mysteries and science fiction.
- 5 purchased science fiction and romance novels.
- 11 purchased mysteries and romance novels.
- 2 purchased all three types of books.

How many of the customers surveyed purchased
a) only mysteries?
b) mysteries and science fiction, but not romance novels?
c) mysteries or science fiction?
d) mysteries or science fiction, but not romance novels?
e) exactly two types?

8. Movies A survey of 350 customers was taken at Regal Cinemas in Austin, Texas, regarding the type of movies customers liked. The following information was determined.
- 196 liked dramas.
- 153 liked comedies.
- 88 liked science fiction.
- 59 liked dramas and comedies.
- 37 liked dramas and science fiction.
- 32 liked comedies and science fiction.
- 21 liked all three types of movies.

Of the customers surveyed, how many liked
a) none of these types of movies?
b) only dramas?
c) exactly one of these types of movies?
d) exactly two of these types of movies?
e) dramas or comedies?

9. Jobs at a Restaurant Panera Bread compiled the following information regarding 30 of its employees. The following was determined.
- 8 cooked food.
- 9 washed dishes.
- 18 operated the cash register.
- 4 cooked food and washed dishes.
- 5 washed dishes and operated the cash register.
- 3 cooked food and operated the cash register.
- 2 did all three jobs.

How many of the employees
a) only cooked food?
b) only operated the cash register?
c) washed dishes and operated the cash register but did not cook food?
d) washed dishes or operated the cash register but did not cook food?
e) did at least two of these jobs?

10. Electronic Devices In a survey of college students, it was found that
- 356 owned an iPod.
- 293 owned a laptop.
- 285 owned a gaming system.
- 193 owned an iPod and a laptop.
- 200 owned an iPod and a gaming system.
- 129 owned a laptop and a gaming system.
- 68 owned an iPod, a laptop, and a gaming system.
- 26 owned none of these devices.

a) How many college students were surveyed?

Of the college students surveyed, how many owned
b) an iPod and a gaming system, but not a laptop?
c) a laptop, but neither an iPod nor a gaming system?
d) exactly two of these devices?
e) at least one of these devices?

11. Homeowners' Insurance Policies A committee of the Florida legislature decided to analyze 350 homeowners' insurance policies to determine if the consumers' homes
were covered for damage due to sinkholes, mold, and floods. The following results were determined.

170 homes were covered for damage due to sinkholes.
172 homes were covered for damage due to mold.
234 homes were covered for damage due to floods.
105 homes were covered for damage due to sinkholes and mold.
115 homes were covered for damage due to mold and floods.
109 homes were covered for damage due to sinkholes and floods.
78 homes were covered for damage due to all three conditions.

How many of the homes

a) were covered for damage due to mold but were not covered for damage due to sinkholes?

b) were covered for damage due to sinkholes or mold?
c) were covered for damage due to mold and floods but were not covered for damage due to sinkholes?

d) were not covered for damage due to any of the three conditions?

12. Appetizers Survey Da Tulio’s Restaurant hired Dennis Goldstein to determine what kind of appetizers customers liked. He surveyed 100 people, with the following results: 78 liked shrimp cocktail, 56 liked mozzarella sticks, and 35 liked both shrimp cocktail and mozzarella sticks. Every person interviewed liked one or the other or both kinds of appetizers. Does this result seem correct? Explain your answer.

25 cars were driven by women and had two or more passengers.
20 cars were driven by U.S. citizens and had two or more passengers.
15 cars were driven by women who are U.S. citizens and had two or more passengers.

After his supervisor reads the report, she explains to the agent that he made a mistake. Explain how his supervisor knew that the agent’s report contained an error.

Challenge Problems/Group Activities

14. Parks A survey of 300 parks showed the following.

15 had only camping.
20 had only hiking trails.
35 had only picnicking.
185 had camping.
140 had camping and hiking trails.
125 had camping and picnicking.
210 had hiking trails.

Determine the number of parks that

a) had at least one of these features.
b) had all three features.
c) did not have any of these features.
d) had exactly two of these features.

15. Surveying Farmers A survey of 500 farmers in a midwestern state showed the following.

125 grew only wheat.
110 grew only corn.
90 grew only oats.
200 grew wheat.
60 grew wheat and corn.
50 grew wheat and oats.
180 grew corn.

Determine the number of farmers who

a) grew at least one of the three.
b) grew all three.
c) did not grow any of the three.
d) grew exactly two of the three.
16. Family Reunion When the Montesano family discussed where their annual reunion should take place, they found that of all the family members:

8 would not go to a park.
7 would not go to a beach.
11 would not go to the family cottage.
3 would go to neither a park nor a beach.
4 would go to neither a beach nor the family cottage.
6 would go to neither a park nor the family cottage.
2 would not go to a park or a beach or to the family cottage.
1 would go to all three places.

What is the total number of family members?

Recreational Mathematics

17. Number of Elements A universal set $U$ consists of 12 elements. If sets $A$, $B$, and $C$ are proper subsets of $U$ and $n(U) = 12$, $n(A \cap B) = n(A \cap C) = n(B \cap C) = 6$, $n(A \cap B \cap C) = 4$, and $n(A \cup B \cup C) = 10$, determine

a) $n(A \cup B)$

b) $n(A' \cup C)$

c) $n(A \cap B')$

SECTION 2.6 Infinite Sets

Which set is larger, the set of integers or the set of even integers? One might argue that because the set of even integers is a subset of the set of integers, the set of integers must be larger than the set of even integers. Yet both sets are infinite sets, so how can we determine which set is larger? This question puzzled mathematicians for centuries until 1874, when Georg Cantor developed a method of determining the cardinal number of an infinite set. In this section, we will discuss infinite sets and how to determine the number of elements in an infinite set.

Why This is Important The concept of infinity and which sets contain more elements has led to the expansion and understanding of many mathematical and scientific concepts.

On page 45, we state that a finite set is a set in which the number of elements is zero or the number of elements can be expressed as a natural number. On page 46, we define a one-to-one correspondence. To determine the number of elements in a finite set, we can place the set in a one-to-one correspondence with a subset of the set of counting numbers. For example, the set $A = \{\#, ?, \$\}$ can be placed in one-to-one correspondence with set $B = \{1, 2, 3\}$, a subset of the set of counting numbers.

$$A = \{\#, ?, \$\}$$
$$\downarrow \downarrow \downarrow$$
$$B = \{1, 2, 3\}$$

Because the cardinal number of set $B$ is 3, the cardinal number of set $A$ is also 3. Any two sets, such as set $A$ and set $B$, that can be placed in a one-to-one correspondence must have the same number of elements (therefore the same cardinality) and must be equivalent sets. Note that $n(A)$ and $n(B)$ both equal 3.

German mathematician Georg Cantor (1845–1918), known as the father of set theory, thought about sets that were not bounded. He called an unbounded set an infinite set and provided the following definition.
Example 5 The Cardinal Number of the Set of Odd Numbers
Show that the set of odd counting numbers has cardinality \( \aleph_0 \).

Solution To show that the set of odd counting numbers has cardinality \( \aleph_0 \), we need to show a one-to-one correspondence between the set of counting numbers and the set of odd counting numbers.

\[
\begin{align*}
\text{Counting numbers:} & \quad N = \{1, 2, 3, 4, 5, \ldots, n, \ldots\} \\
\text{Odd counting numbers:} & \quad O = \{1, 3, 5, 7, \ldots, 2n - 1, \ldots\}
\end{align*}
\]

Since there is a one-to-one correspondence, the odd counting numbers have cardinality \( \aleph_0 \); that is, \( n(O) = \aleph_0 \).

We have shown that both the odd and the even counting numbers have cardinality \( \aleph_0 \). Merging the odd counting numbers with the even counting numbers gives the set of counting numbers, and we may reason that

\[
\aleph_0 + \aleph_0 = \aleph_0
\]

This result may seem strange, but it is true. What could such a statement mean? Well, consider a hotel with infinitely many rooms. If all the rooms are occupied, the hotel is, of course, full. If more guests appear, wanting accommodations, will they be turned away? The answer is no, for if the room clerk were to reassign each guest to a new room with a room number twice that of the present room, all the odd-numbered rooms would become unoccupied and there would be space for infinitely many more guests!

Cantor showed that there are different orders of infinity. Sets that are countable and have cardinal number \( \aleph_0 \) are the lowest order of infinity. Cantor showed that the set of integers and the set of rational numbers (fractions of the form \( p/q \), where \( q \neq 0 \)) are infinite sets with cardinality \( \aleph_0 \). He also showed that the set of real numbers (discussed in Chapter 5) could not be placed in a one-to-one correspondence with the set of counting numbers and that they have a higher order of infinity.

SECTION 2.6 Exercises

Warm Up Exercises
In exercises 1–2, fill in the blank with an appropriate word, phrase, or symbol(s).

1. A set that can be placed in a one-to-one correspondence with a proper subset of itself is called a(n) ________ set.
2. A set that is finite or can be placed in a one-to-one correspondence with the set of counting numbers is called a(n) ________ set.

Practice the Skills
In Exercises 3–12, show that the set is infinite by placing it in a one-to-one correspondence with a proper subset of itself. Be sure to show the pairing of the general terms in the sets.

3. \( \{3, 4, 5, 6, 7, \ldots\} \)
4. \( \{30, 31, 32, 33, 34, \ldots\} \)
5. \( \{3, 5, 7, 9, 11, \ldots\} \)
6. \( \{20, 22, 24, 26, 28, \ldots\} \)
7. \( \{5, 9, 13, 17, 21, \ldots\} \)
8. \( \{6, 11, 16, 21, 26, \ldots\} \)
9. \( \left\{ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots \right\} \)
10. \( \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \)
11. \( \left\{ \frac{4}{11}, \frac{5}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \ldots \right\} \)
In Exercises 13–22, show that the set has cardinal number \( \aleph_0 \) by establishing a one-to-one correspondence between the set of counting numbers and the given set. Be sure to show the pairing of the general terms in the sets.

13. \{3, 6, 9, 12, 15, \ldots \}  
14. \{40, 41, 42, 43, 44, \ldots \}
15. \{4, 8, 12, 16, 20, \ldots \}  
16. \{0, 2, 4, 6, 8, \ldots \}
17. \{2, 5, 8, 11, 14, \ldots \}  
18. \{7, 11, 15, 19, 23, \ldots \}
19. \\{\frac{3}{3}, \frac{6}{6}, \frac{9}{9}, \frac{12}{12}, \frac{15}{15}, \ldots \}\  
20. \\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \ldots \}
21. \\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \ldots \}\  
22. \\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \}

Challenge Problems/Group Activities

In Exercises 23–26, show that the set has cardinal number \( \aleph_0 \) by establishing a one-to-one correspondence between the set of counting numbers and the given set.

23. \{1, 4, 9, 16, 25, \ldots \}  
24. \{2, 4, 8, 16, 32, \ldots \}
25. \{3, 9, 27, 81, 243, \ldots \}  
26. \{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \ldots \}

Recreational Mathematics

In Exercises 27–31, insert the symbol \(<\), \(>,\) or \(=\) in the shaded area to make a true statement.

27. \( \aleph_0 \quad \aleph_0 + \aleph_0 \)  
28. \( 2\aleph_0 \quad \aleph_0 + \aleph_0 \)
29. \( 2\aleph_0 \quad \aleph_0 \)  
30. \( \aleph_0 + 5 \quad \aleph_0 + 3 \)
31. \( n(\mathbb{N}) \quad \aleph_0 \)

32. There are a number of paradoxes (a statement that appears to be true and false at the same time) associated with infinite sets and the concept of infinity. One of these, called Zeno's Paradox, is named after the mathematician Zeno, born about 496 B.C. in Italy. According to Zeno's paradox, suppose Achilles starts out 1 meter behind a tortoise. Also, suppose Achilles walks 10 times as fast as the tortoise crawls. When Achilles reaches the point where the tortoise started, the tortoise is \( \frac{1}{10} \) of a meter ahead of Achilles; when Achilles reaches the point where the tortoise was \( \frac{1}{10} \) of a meter ahead, the tortoise is now \( \frac{1}{100} \) of a meter ahead; and so on. According to Zeno's Paradox, Achilles gets closer and closer to the tortoise but never catches up to the tortoise.

a) Do you believe the reasoning process is sound? If not, explain why not.

b) In actuality, if this situation were real, would Achilles ever pass the tortoise?

Internet/Research Activities

33. Do research to explain how Cantor proved that the set of rational numbers has cardinal number \( \aleph_0 \).

34. Do research to explain how it can be shown that the real numbers do not have cardinal number \( \aleph_0 \).

**CHAPTER 2 Summary**

**Important Facts and Concepts**

<table>
<thead>
<tr>
<th>Section 2.1</th>
<th>Methods Used to Indicate a Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
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<tr>
<td>Roster Form</td>
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<td>Set-Builder Notation</td>
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<table>
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<tr>
<th>Symbol</th>
<th>Meaning</th>
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<td>( \in )</td>
<td>is an element of</td>
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<tr>
<td>( \notin )</td>
<td>is not an element of</td>
</tr>
<tr>
<td>( n(A) )</td>
<td>number of elements in set ( A )</td>
</tr>
<tr>
<td>( \emptyset ) or ( { } )</td>
<td>the empty set</td>
</tr>
<tr>
<td>( U )</td>
<td>the universal set</td>
</tr>
</tbody>
</table>

**Examples and Discussion**

| Example 1, page 43 |
| Examples 2–3, 5–7 pages 44, 45 |
| Examples 4–6, pages 44–45 |

Example 4–6, pages 44–45 |