Linear Programming: Simplex Method

5.1. The Simplex Tableau; Pivoting

In this section we will learn how to prepare a linear programming problem in order to solve it by pivoting using a matrix method. The Simplex Method is matrix based method used for solving linear programming problems with any number of variables. The simplex algorithm can be used to solve linear programming problems that already are, or can be converted to, standard maximum-type problems. An example of a standard maximum-type problem is

Maximize $P = 4x + 4y$

subject to $x + 3y \leq 30$

$2x + y \leq 20$

$x \geq 0; y \geq 0$
The Standard Maximum-Type Problem

A linear programming problem is a standard maximum-type problem if the following conditions are met:

- The objective function is linear and is to be maximized.
- The variables are all nonnegative.
- The structural constraints are all of the form $ax + by + \cdots \leq c$, where $c \geq 0$.

Exercise 176. (i) Determine which of the constraints below are not on the form appropriate for a standard maximum-type problem. (ii) If the constraint is not on the appropriate form, rewrite it if possible, to put on that form.

(a) $3x - 2y \leq 6$
(b) $-6x + 5y + z \geq -3$
(c) $7x - 4y \leq -5$
(d) $x - y - 11z \leq 0$
(e) $5x - 7y \geq -2$
(f) $3x - 5z \leq -8$
(g) $2x \leq 50$
Exercise 177. Which ones of the following linear programming problems are standard maximum-type problems? Please motivate your answer.

(a) 
Maximize \( P = 4x + 4y \)
subject to \( x + 3y \leq 30 \)
\( 2x + y \leq 20 \)
\( x \geq 0; y \geq 0 \)

(b) 
Minimize \( C = 5x + 3y \)
subject to \( x + y \geq 6 \)
\( 6x + y \geq 16 \)
\( x + 6y \geq 16 \)
\( x \geq 0; y \geq 0 \)
The first step towards using the simplex algorithm is to convert the structural constraints into equalities by adding so called \textbf{slack variables}. The following structural constraints,

\[
x + 2y \leq 8, \\
x - y \leq 4,
\]

turn into the following equalities or \textbf{slack equations} when slack variables, \( s_1 \) and \( s_2 \) are added to the first and second constraint, respectively:

\[
x + 2y + s_1 = 8, \\
x - y + s_2 = 4.
\]

The slack variables will always be nonnegative (zero or positive) when solving linear programming problems. In a linear programming problem with two variables, the slack variables are always nonnegative in the corner points of the feasible region. Should a slack variable turn negative, it indicates that a mistake has been made in the pivoting. For a more in depth analysis of how the slack variables relate to the geometry of the feasible region, the reader should refer to section 4.1 in the textbook. The slack variables are merely a tool to help us solve linear programming problems. When stating the solution of the problem, we will not state the values of the slack variables, only the values of the variables given in the problem.
The next step is to insert the slack equations into an augmented matrix.

\[
\begin{bmatrix}
x & y & s_1 & s_2 \\
1 & 2 & 1 & 0 & 8 \\
1 & -1 & 0 & 1 & 4 \\
\end{bmatrix}
\]

At a stage in the pivoting process, after a pivot operation has been completed, we can determine the values of all our variables \((x, y, s_1, \text{ and } s_2)\). Before we begin pivoting, we notice that \(x\) and \(y\) are free variables which we can set to any value. In the simplex algorithm we call our free variables \textbf{nonbasic variables} and we set them equal to zero. The other variables, whose column contains exactly one 1 and the rest of the elements are zero, are called the \textbf{basic variables}. Solving the two equations from the augmented matrix,

\[
x + 2y + s_1 = 8,
\]

\[
x - y + s_2 = 4.
\]

for the basic variables, \(s_1\) and \(s_2\), we obtain,

\[
s_1 = 8 - x - 2y,
\]

\[
s_2 = 4 - x + y.
\]
and since we set the nonbasic variables equal to zero, the values of our variables before we begin pivoting are:

\[
\begin{align*}
x &= 0 \\
y &= 0 \\
s_1 &= 8 \\
s_2 &= 4.
\end{align*}
\]

**The Smallest-Quotient Rule**

For any given pivot column,

- Divide each positive number in that column into the corresponding number in the constants column of the matrix.
- Select as the pivot row, the row corresponding to the smallest nonnegative quotient obtained. If a zero quotient is obtained, it is the smallest quotient.
- Pivot using the pivot element, making it equal to 1 and all other elements in that column equal to zero.
- When pivoting is done, and we set the nonbasic variables to zero, we obtain a solution called a basic feasible solution to the linear programming problem.

Using the smallest quotient rule to choose the pivot element in a given matrix ensures that the slack variables remain non-negative. In a linear programming problem with two variables, the basic feasible solution corresponds to a corner point of the feasible region.
Exercise 178. Using the given augmented matrix below,

\[
\begin{bmatrix}
1 & 2 & 1 & 0 & 8 \\
1 & -1 & 0 & 1 & 4
\end{bmatrix}
\]

(a) Use the smallest-quotient rule to find the pivot element in the \(x\) column.

(b) Perform the pivot operation, making the pivot element equal to 1, and all other elements in that column equal to zero.

(c) Identify the nonbasic and the basic variables. Set the nonbasic variables equal to zero and find the basic feasible solution.
5.2. The Simplex Method: Solving Maximum Problems in Standard Form

In the previous section we learned to identify a standard maximum-type linear programming problem, how to add slack variables to the structural constraints, to set up the augmented matrix, given a pivot column apply the smallest quotient rule to find the pivot element, and once the pivot operation has been completed, find the basic feasible solution. But where does the objective function come in? How do we know which column to choose as the pivot column? How do we find the maximum value of the objective function? We are now ready to learn all the steps in the Simplex Algorithm. All those questions will be answered next.
Consider the following standard maximum-type linear programming problem.

Maximize \( P = 3x + 4y \)
subject to \( x + 3y \leq 30 \)
\( 2x + y \leq 20 \)
\( x \geq 0; y \geq 0 \)

**Step 1 in the Simplex Algorithm - Insert Slack Variables**
Insert a slack variable into each of the structural constraints. The result is this system of slack equations:

\[
\begin{align*}
  x + 3y + s_1 &= 30, \\
  2x + y + s_2 &= 20.
\end{align*}
\]

**Step 2 in the Simplex Algorithm - Rewrite the Objective Function**
Rewrite the objective function to match the format of the slack equations, and add the corresponding equation to the bottom of the slack equations:

\[
\begin{align*}
  x + 3y + s_1 &= 30, \\
  2x + y + s_2 &= 20, \\
  -3x - 4y + P &= 0.
\end{align*}
\]
Step 3 in the Simplex Algorithm - Write the Initial Simplex Tableau

The augmented matrix is called the initial simplex tableau. Each number in the bottom row, to the left of the vertical bar is called an indicator.

\[
\begin{bmatrix}
1 & 3 & 1 & 0 & 0 & | & 30 \\
2 & 1 & 0 & 1 & 0 & | & 20 \\
-3 & -4 & 0 & 0 & 1 & | & 0 \\
\end{bmatrix}
\]

Step 4 in the Simplex Algorithm - Find the Pivot Element

The most negative indicator in the last row of the tableau determines the pivot column. We apply the smallest quotient rule for that column. Find the pivot element of our initial simplex tableau below.

\[
\begin{bmatrix}
1 & 3 & 1 & 0 & 0 & | & 30 \\
2 & 1 & 0 & 1 & 0 & | & 20 \\
-3 & -4 & 0 & 0 & 1 & | & 0 \\
\end{bmatrix}
\]
Step 5 in the Simplex Algorithm - Perform the Pivot Operation

Perform the pivot operation on the pivot element. After the pivot operation has been completed write down the basic feasible solution.

\[
\begin{bmatrix}
1 & 3 & 1 & 0 & 0 & | & 30 \\
2 & 1 & 0 & 1 & 0 & | & 20 \\
-3 & -4 & 0 & 0 & 1 & | & 0 \\
\end{bmatrix}
\]

\[
x = _____, \ y = _____, \ s_1 = _____, \ s_2 = _____ \\
P = _____
\]

Step 6 in the Simplex Algorithm

If a negative indicator is still present, repeat steps 4 and 5. If no negative indicators are present, the maximum of the objective function has been reached.

\[
\begin{bmatrix}
\frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & | & 10 \\
\frac{5}{3} & 0 & -\frac{1}{3} & 1 & 0 & | & 10 \\
-\frac{5}{3} & 0 & \frac{4}{3} & 0 & 1 & | & 40 \\
\end{bmatrix}
\]
We had to repeat steps 4 and 5 since we still did have a negative indicator in the last row. The matrix obtained after the last pivoting is listed below. Are there any more negative indicators? What is the basic feasible solution after the last pivoting?

\[
\begin{bmatrix}
0 & 1 & \frac{2}{5} & -\frac{1}{5} & 0 & & 8 \\
1 & 0 & -\frac{1}{5} & \frac{3}{5} & 0 & & 6 \\
0 & 0 & 1 & \frac{3}{5} & 1 & & 50 \\
\end{bmatrix}
\]

\[x = _____, \ y = _____, \ s_1 = _____, \ s_2 = _____, \ P = _____\]

The solution to our linear programming problem is: \(P\) reaches a maximum value of 50 for \(x = 6\) and \(y = 8\). Notice that we do not even mention the slack variables. They are not part of the final solution, but just a tool to help us solve the problem.
5.2. The Simplex Method: Solving Maximum Problems in Standard Form

The Simplex Algorithm Flowchart

1. Insert slack variables and find slack equations
2. Rewrite the objective function and put it below the slack equations
3. Write the initial simplex tableau
4. Find the pivot element by finding the most negative indicator in last row and using the smallest quotient rule.
5. Perform the pivot operation.
6. Are there any more negative indicators in the last row?
   - yes
   - no, The maximum has been reached.
Exercise 179. Use the simplex method to solve the following linear programming problem. After each pivot operation, list the basic feasible solution. You final answer should be $f_{\text{max}}$ and the $x$- and $y$-values for which $f$ assumes its maximum value.

Maximize $f = 2x + y$
subject to $x + 3y \leq 14$
$\quad 2x + y \leq 11$
$\quad x \geq 0; \quad y \geq 0$
Exercise 180. Use the simplex method to solve the following linear programming problem. After each pivot operation, list the basic feasible solution. You final answer should be $f_{\text{max}}$ and the $x$-, $y$-, and $z$-values for which $f$ assumes its maximum value.

Maximize $f = 2x + y + 3z$

subject to $x + 2y + z \leq 25$

$3x + 2y + 2z \leq 30$

$x \geq 0; y \geq 0; z \geq 0$
Exercise 181. Find the objective function and the constraints, and then solve the problem by using the simplex method.

A confectioner has 600 pounds of chocolate, 100 pounds of nuts, and 50 pounds of fruits in inventory with which to make three types of candy: Sweet Tooth, Sugar Dandy, and Dandy Delite. A box of Sweet Tooth uses 3 pounds of chocolate, 1 pound of nuts, and 1 pound of fruit and sells for $8. A box of Sugar Dandy requires 4 pounds of chocolate and \( \frac{1}{2} \) pound of nuts and sells for $5. A box of Dandy Delite requires 5 pounds of chocolate, \( \frac{3}{4} \) pounds of nuts, and 1 pound of fruit and sells for $6. How many boxes of each type of candy should be made from the available inventory to maximize revenue?