CHAPTER 8
Additional Probability Topics

8.1. Conditional Probability

Conditional probability arises in probability experiments when the person performing the experiment is given some extra information about the outcome.

For example, if you collected data on the rates of lung cancer in the United States. Suppose you wanted to find the probability that a randomly selected person has lung cancer. What would happen to the probability if you were given some extra information about the person, that he was a smoker. Would that affect the probability that he had lung cancer? In which way?

Suppose that someone rolls a single die out of sight, and tells you it came up with an odd number. You are then asked, ”What is the probability that a 3 has been rolled?”. This extra information reduced the number of possible outcomes from 6 outcomes: \( S = \{1, 2, 3, 4, 5, 6, \} \) to 3 outcomes: \( S^* = \{1, 3, 5\} \), where \( S^* \) is the reduced
sample space. The probability of rolling a 3, **given it was an odd number** is therefore $\frac{1}{3}$, or more formally:

$$P(\text{a 3 comes up}|\text{an odd number has been rolled}) = \frac{1}{3},$$

where the vertical bar is read ”given that” and the event to the right is the condition that is given. For events $A$ and $B$, $P(A|B)$ is read ”the probability of $A$, given that $B$ has already occurred.

**Exercise 69.** Suppose a population of 500 people includes 30 teachers and 240 females. There are 24 females who are teachers. A person is chosen at random, and we are told the person is a female. Find the probability that the person is a teacher, given it was a female. Hint, let the reduced sample space $S^* = F$, where $F$ is the event that the person was a female. In the reduced sample space, divide the number of female teachers by the number of females.
Conditional Probability
If $E$ and $F$ be events of a sample space $S$, and suppose $P(F) > 0$. The **conditional probability of the event** $E$, **assuming event** $F$ denoted by $P(E|F)$ is defined as:

\begin{equation}
P(E|F) = \frac{P(E \cap F)}{P(F)}
\end{equation}

Conditional Probability - Equally Likely Outcomes
If $E$ and $F$ be events of a sample space $S$ for which each outcome is equally likely, and suppose $P(F) > 0$. The **conditional probability of the event** $E$, **assuming event** $F$ denoted by $P(E|F)$ is defined as:

\begin{equation}
P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{n(E \cap F)}{n(F)}
\end{equation}

**Exercise 70.** If two cards are randomly drawn, in succession, without replacement, from a deck of 52 cards,

(a) what is the probability that the second card is a heart, given that first card was a heart?

(b) what is the probability that the second card is a queen, given that the first card was a queen?
Exercise 71. Suppose $E$ and $F$ are events of a sample space for which $P(E) = 0.5$, $P(F) = 0.8$, and $P(E \cap F) = 0.4$. Find

(a) $P(E \cup F)$

(b) $P(E|F)$

(c) $P(F|E)$

(d) $P(\overline{E}|\overline{F})$

Exercise 72. If three balls are randomly drawn, in succession and without replacement, from a box containing five red and seven green balls. What is the probability that the third ball drawn is red, given that the first two balls were green? Draw a picture of the box before the first draw, second draw and third draw.
Exercise 73. The following data were collected from a finite mathematics class at State University.

<table>
<thead>
<tr>
<th></th>
<th>Have a Scholarship</th>
<th>No Scholarship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freshman</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>Sophomore</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>Junior</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Let $E$ be the event, ”A person has a scholarship” and let $F$ be the event, ”A person is a freshman”. What is the probability that the student chosen has a scholarship, given that the person is a freshman? Or in other words, find $P(E|F)$. 
Exercise 74. Three slips of paper with a 1, 2 and 3 written on them respectively are placed in a box. Two slips are randomly drawn, with replacement, and the first and second number drawn is recorded.

(a) List the sample space for this experiment.
\[ S = \{ \}

(b) Find the probability that the sum is five.

(c) Find the probability that the sum is five and the first number is 3.

(d) Use the information above and the conditional probability formula to find the probability that the first number is a 3, given that the sum is 5.

(e) Find the probability that the first number is a 3, given that the sum is 5, by using the reduced sample space, \( S^* \).
\[ S^* = \{ \} \]
Exercise 75. If \( P(A|B) = \frac{2}{3} \) and \( P(B) = \frac{5}{8} \), find \( P(A \cap B) \).

Exercise 76. Suppose that two balls are randomly drawn, in succession and without replacement, from a box containing five red and seven green balls.

(a) Draw and label a tree diagram that will describe the probabilities of the various outcomes.

(b) Find the probability that the first ball is red and the second ball is red, i.e. \( P(1st \ R \ and \ 2nd \ R) \).

(c) Find the probability that the first ball is green and the second ball is red, i.e. \( P(1st \ G \ and \ 2nd \ R) \).
8.2. Independent Events.

In this section, we focus on finding the probability of the intersection of events. We will derive a formula for the intersection of two events from the conditional probability formula.

Exercise 77. Suppose a sample of two computers is randomly taken from a container with 5 defective computers and 11 working computers. What is the probability that the first computer selected is good and the second computer selected is defective? Draw a probability tree for this experiment.
The conditional probability formula from last section, for \( P(A|B) \) is

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}.
\]

If we solve this equation for \( P(A \cap B) \), we obtain:

1. \( P(A \cap B) = P(B)P(A|B) \)

We also have the equation for \( P(B|A) \):

\[
P(B|A) = \frac{P(A \cap B)}{P(A)}.
\]

If we solve this equation for \( P(A \cap B) \), we obtain:

2. \( P(A \cap B) = P(A)P(B|A) \)

If we put these two equations together, we obtain the Multiplication Rule for the Intersection of Events:

**Multiplication Rule for the Intersection of Two Events**

For any two events \( A \) and \( B \), in a sample space \( S \), with \( P(A) \neq 0 \) and \( P(B) \neq 0 \), we have

\[
(8.3) \quad P(A \text{ and } B) = P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)
\]

**Exercise 78.** Two cards are to be randomly selected, in succession, without replacement, from a deck of 52 cards. What is the probability that the first card will be a diamond and the second card will be a club?
From the formula above, you could either find
\( P( \text{1st club and 2nd diamond} ) \) from the product
\[
P( \text{1st diamond} ) P( \text{2nd club | 1st diamond} ),
\]
or from the product
\[
P( \text{2nd club} ) P( \text{1st diamond | 2nd club} ),
\]
however, we chose the first form of the equation as it comes
more naturally. In the next section we will cover the case
when we have "backwards" conditional probability, i.e. when
the condition is an event which happened later in time.

The multiplication rule for the intersection of events can
be extended to include several events:

**Multiplication Rules for the Intersection of Several
Events**
The multiplication rule can be extended to several events
as follows:

\[
(P(A \cap B \cap C \cap D \ldots) = P(A) \cdot P(B | A) \cdot P(C | A \cap B) \cdot P(D | A \cap B \cap C) \ldots
\]
Exercise 79. Three cards are to be randomly selected, in succession, without replacement, from a deck of 52 cards. What is the probability that the first card will be a diamond and the second diamond and the third card will be a club?

Sometimes there is no “natural order” to the two events involved:

Exercise 80. Research by a department store revealed that 80% of the customers are women, and that 75% of those women’s purchases are charged on the chain’s credit cards. In addition, 35% of the male customers’ purchases are charged on the chain’s credit cards.

(a) Draw a tree diagram for these data.

(b) What is the probability that a person making purchase form this chain is a woman and charge her purchase on her credit card.
Independent Events
Two events $A$ and $B$ are independent if the occurrence of one has no effect on the probability of the other occurring. Thus,

\begin{align}
P(A|B) &= P(A) \\
\text{and} \\
P(B|A) &= P(B)
\end{align}

Here are some examples of independent and dependent events:

<table>
<thead>
<tr>
<th>Independent Events</th>
<th>Dependent Events</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draws of card with replacements</td>
<td>Draws of card w/o replacements</td>
</tr>
<tr>
<td>Draws of marbles with replacement</td>
<td>Draws of marbles w/o replacement</td>
</tr>
<tr>
<td>Tosses of a coin</td>
<td>The weather tomorrow and the weather today</td>
</tr>
<tr>
<td>Repeated rolls of a die</td>
<td></td>
</tr>
</tbody>
</table>
The Multiplication Rule for Independent Events

If $A$ and $B$ are independent events in a sample space, then

\begin{equation}
P(A \cap B) = P(A) \cdot P(B)
\end{equation}

Exercise 81. A single die is rolled twice. What is the probability that the first roll is a 3 and the second roll is a 5?

Exercise 82. Nuclear power plants have a threefold security system, each of which is 98% reliable and independent of the others, to prevent unauthorized persons from entering the premises. What is the probability that an unauthorized person will

(a) get through all three security systems.

(b) get through the first two systems, but not the third.
8.3. Bayes’ Theorem

Bayes’ theorem is a special application of conditional probability, (i) when the event in the condition occurs after the event whose probability we are calculating, or (ii) the event in the condition occurs further out in the probability tree diagram than the event whose probability we are calculating. We will illustrate this in the next problem.
Exer cise 83. Surf Mart, which sells shirts under its own label buys 40% of its shirts from supplier $A$, 50% from supplier $B$, and 10% from supplier $C$. It is found that 2% of the shirts from $A$ have flaws, 3% from $B$ have flaws, and 5% from $C$ have flaws. A probability tree diagram representing these purchases and flaw rates is shown below. If one of these shirts of bought from Surf Mart,

(a) what is the probability that the shirt has a flaw, given that it came from $B$?

(b) what is the probability that the shirt has a flaw?

(c) what is the probability that the shirt came from $B$, given that is has a flaw?
From the Venn diagram below, we see that the sample space is divided into three mutually exclusive events, $A$, $B$, and $C$. Notice that the event $F$ is the union of three mutually exclusive events: $A \cap F$, $B \cap F$, and $C \cap F$. Therefore we found $P(F)$ by adding $P(A \cap F) + P(B \cap F) + P(C \cap F)$.

![Venn Diagram]

We calculated the "backwards" conditional probability by using the formula:

$$P(B|F) = \frac{P(B \cap F)}{P(F)} = \frac{P(B \cap F)}{P(A \cap F) + P(B \cap F) + P(C \cap F)}$$

This is a form of Bayes’ theorem stated below:

**Bayes’ Theorem**

Let $A$ and $B$ be mutually exclusive events which make up the whole sample space, i.e. $A \cup B = S$. Let $F$ be any event whose probability is not zero. Then,

(8.8) $P(A|F) = \frac{P(A \cap F)}{P(F)} = \frac{P(A \cap F)}{P(A \cap F) + P(B \cap F)}$

(8.9) $= \frac{P(A)P(F|A)}{P(A)P(F|A) + P(B)P(F|B)}$
A more general form of Bayes’ theorem is listed in the book for the case when the sample space is divided into many mutually exclusive events, $A_1, A_2, ..., A_n$.

**Exercise 84.** Use the tree diagram below to find the following probabilities.

(a) $P(D|A)$

(b) $P(A \cap D)$

(c) $P(D)$

(d) $P(A|D)$
Exercise 85. Records indicate that 2% of the population has a certain kind of cancer. A medical test has been devised to help detect this kind of cancer. If a person does have the cancer, the test will detect it 98% of the time. However, 3% of the time the test will indicate that a person has the cancer when, in fact, he or she does not. For persons using this test, what is the probability that

(a) the person has this type of cancer and the test indicates that he or she has it?

(b) the person has this type of cancer, given that the test indicates that he or she has it?

(c) the person does not have this type of cancer, given a positive result for it?
8.4. Permutations

8.4.1. Factorials

Counting problems often involve the product of consecutive numbers. To save on the amount of writing, we use the factorial notation. For example, 3! is read ”three factorial” and is defined by:

\[ 3! = 1 \cdot 2 \cdot 3 = 6, \text{ and} \]
\[ 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720. \]

**n Factorial**

Let \( n \) be a positive integer. Then the product of integers from 1 to \( n \), \( n! \), read ”\( n \) factorial” is:

\[ n! = 1 \cdot 2 \cdot 3 \cdots n. \]

By definition, 0! = 1, to make all calculations work out properly.

**Exercise 86.** Use your calculator to find:

(a) 10!

(b) 20!

(c) 50!

(d) 100!

You will notice how quickly factorials grow big.
Permutations - ORDER IS IMPORTANT

A permutation is an ordered arrangement of objects for which:

- All objects are selected from the same set, $S$.
- All objects are considered distinguishable, i.e. we can tell them apart.
- Successive selections from $S$ are made without replacement.

The result is called an **ordered arrangement**.

**Exercise 87.** In how many ways can three out of seven executives be seated in a row for a corporate picture?

In the previous example, we name the number of permutations (ordered arrangements) of three people, selected from a group of seven people, $P(7, 3)$.

In general, if we want to find the number of permutations of $n$ distinguishable objects taken $r$ at a time, we obtain the following formula:

**The Number of Permutations of $n$ Distinguishable Objects Taken $r$ at a Time** where $0 \leq r \leq n$.

\[
P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot (n - r + 1)
\]

\[
= \frac{n!}{(n - r)!}
\]
On your TI-83 or TI-84 calculator, you can find factorial (!) and $P(n, r)$ ($nP$ on your calculator) on the MATH menu by pressing MATH→PRB.

To find $35!$, press: $35 \rightarrow$ MATH→PRB→!
To find $P(7, 4)$, press: $7 \rightarrow$ MATH→PRB→ $nP$→ 4

**Exercise 88.** In how many ways can three people be elected president, treasurer and secretary, in a chess club with 22 members?

**Exercise 89.** In how many ways can we arrange 3 red books, 1 blue book and 1 green book on a shelf?

Solution: An example of an arrangement of the books is: $RBRRG$.
Now imagine that we label each of the red books with a number inside the cover: $R_1, R_2,$ and $R_3$, making the red books distinguishable. In the table below, we list all possible arrangements of $RBRRG$ when the red books are indistinguishable (they all look the same) versus when they are distinguishable (they each are labeled with a different number):
1. Indistinguishable  
2. Distinguishable

\[ RBRRG \]
\[ R_1 BR_2 R_3 G \]
\[ R_1 BR_3 R_2 G \]
\[ R_2 BR_1 R_3 G \]
\[ R_2 BR_3 R_1 G \]
\[ R_3 BR_1 R_2 G \]
\[ R_3 BR_2 R_1 G \]

For the arrangement \[ RBRRG \], three positions on the bookshelf are taken up by the red books. There are \( 3! = 6 \) ways of lining up the red books (see table), but they all look the same to us and this is the case for any arrangement of the three red, one blue and one green book. There are a total of \( 5! \) permutations of the five books, but for each permutation with three positions of the red books fixed, there are \( 3! \) ways for the red books to be lined up, all of which would look the same to us. Remember that the red books really are indistinguishable to us. (We just pretended they weren’t for the sake of demonstrating all possible arrangements.) Therefore, the number of distinguishable arrangements of the five books is:

\[
\frac{5!}{3!} = \frac{120}{6} = 20
\]
Number of Distinguishable Arrangements with Indistinguishable Objects

Let $S$ be a set of $n$ elements, and let

\[ k_1 = \text{the number of elements of type 1} \]
\[ k_2 = \text{the number of elements of type 2} \]
\[ k_3 = \text{the number of elements of type 3} \]
\[ \vdots \]
\[ k_m = \text{the number of elements of type } m \]

Then the number of distinguishable permutations of the $n$ elements taken $n$ at a time is:

\[
\frac{n!}{k_1!k_2!k_3! \cdots k_m!}
\]

(8.12)

Exercise 90. How many permutations are there of the letters in the word INTELLIGIBLE?

Exercise 91. In how many ways can three people be elected president, treasurer and secretary, in a chess club with 22 members (9 female and 13 male) if at least one of the positions needs to be filled by a female?
Exercise 92. A firm has 750 employees. Explain why at least 2 of the employees would have the same pair of initials for their first and last name.

Exercise 93. For an experiment, 12 sociology students are to be divided into two groups, one containing 7 students and the other containing 5 students. In how many ways can this grouping be done?
8.5. Combinations

Consider the set $A = \{a, b, c, d\}$. How many subsets of three elements can be formed? Recall that when it comes sets and/or subsets, order of elements is not important. The subsets are:

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}.$$

This is a combination problem. From this example we see that $C(4, 3)$, the number of combinations from a set of $n = 4$ elements from which we choose $r = 3$ elements is 4, i.e. $C(4, 3) = 4$

Another example of combinations is card hands. How many 5-card hands are there total? How many of these hands will have all non-face cards? It turns out that there are $C(52, 5) = 2,598,960$, 5-card hands, and that $C(40, 5) = 658,008$ of those hands do not have a face card.
Combinations - ORDER IS NOT IMPORTANT

A combination is a group of objects for which:

- All objects are selected from the same set, \( S \).
- All objects are considered distinguishable, i.e. we can tell them apart.
- Successive selections from \( S \) are made without replacement.
- The order in which they are chosen does not matter.

The result is called a combination, subset or group.

Going back to the example of finding the 3-element subsets of \( A = \{a, b, c, d\} \), how does the number of 3-element subsets of \( A \), \( C(4, 3) \) relate to the number of permutations of the elements of \( A \), \( P(4, 3) \). In the table below compare the number of outcomes for two experiments. In one we randomly choose three letters out of four, and we are not concerned with the order in which they were chosen. In the other experiment, we choose 3 elements from the same set, but here order is important.

<table>
<thead>
<tr>
<th>1. Outcomes in ( C(4, 3) )</th>
<th>2. Outcomes in ( P(4, 3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {a, b, c} )</td>
<td>( abc, acb, bac, bca, cab, cba )</td>
</tr>
<tr>
<td>( {a, b, d} )</td>
<td>( abd, adb, bad, bda, dab, dba )</td>
</tr>
<tr>
<td>( {a, c, d} )</td>
<td>( acd, adc, cad, cda, dac, dca )</td>
</tr>
<tr>
<td>( {b, c, d} )</td>
<td>( bcd, bdc, cbd, cdb, dbc, dc )</td>
</tr>
</tbody>
</table>

We note that for each subset or combination of the letters, there are \( 6 = 3! \) permutations, so \( C(4, 3) \cdot 3! = P(4, 3) \).
In general, if we choose $r$ elements from a set of $n$ elements, for each combination of $r$ elements, there are $r!$ permutations that are counted in $P(n, r)$ but they are not counted in $C(n, r)$. Therefore,

$$C(n, r) \cdot r! = P(n, r)$$

and we have the following formula for combinations:

**The Number of Combinations of $n$ Distinguishable Objects Taken $r$ at a Time** where $0 \leq r \leq n$.

(8.13) \[ C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n - r)!} \]

**Exercise 94.** Use the formulas above to find following number of combinations:

(a) $C(8, 3)$

(b) $C(7, 4)$

On your TI-83 or TI-84 calculator, you can find factorial $C(n, r)$ ($nCr$ on your calculator) on the MATH menu by pressing MATH→PRB.

Redo the exercise above using the $nCr$ function on your calculator.
Exercise 95. How many doubles tennis teams can be formed from 12 players?

Exercise 96. Among 18 computers, 12 are in working order. How many samples of 4 are possible, wherein

(a) all are in working order?

(b) exactly 2 are in working order?

(c) at least 1 is in working order?
Exercise 97. In how many ways can a 4-card hand be dealt if
(a) all if the cards in the hand are to be red cards?

(b) all are to be nines?

(c) all are to be from the same suit?
Exercise 98. A committee of four is to be selected from among eight graduate students and a professor. The committee is to meet with the dean about new classroom equipment. In how many ways can the committee be selected if

(a) there are no restrictions?

(b) the professor must be in the committee?
8.5.1. Probability Using Counting Techniques

**Exercise 99.** A student loan administrator distributes pin numbers to its debtors. Each pin consists of two letters followed by three numbers. (Assuming repetition is allowed and order is important.)

(a) How many different pin numbers are there?

(b) What is the probability that a number selected at random ends in 000?

**Exercise 100.** Each week, eight persons contribute $10.00 to a pool. Every Friday, one name is drawn out of a hat containing the eight names and the winner receives the $80.00.

(a) What is the probability that the same person wins three weeks in a row?

(b) What is the probability that a particular person does not win in 5 weeks?
(c) What is the probability that 5 different people win in the next 5 weeks?

Exercise 101. Through a mix-up on the production line, 6 defective refrigerators were shipped out with 44 good ones. If 5 are selected at random,

(a) what is the probability that all 5 of them are defective?

(b) what is the probability that at least 2 of them are defective?