**Instructions:** Please show your work in the space provided and clearly mark your answers. Remember to include units where appropriate and simplify all answers, unless otherwise stated.

1. (7 pts) Show that the following limit does not exist.

   \[ \lim_{{(x,y) \to (0,0)}} \frac{-2x^4 y e^y}{5x^8 + 8y^2} \]

   - Along \( y = mx \):
     \[ \lim_{{(x, mx) \to (0,0)}} \frac{-2mx^4 e^{mx}}{5x^8 + 8m^2 x^2} = 0 \]
   - Along \( x = x^4 \):
     \[ \lim_{{(x^4, y) \to (0,0)}} \frac{-2x^4 e^{x^4}}{5x^8 + 8x^8} = \frac{-2}{13} \]

   Different limits so limit DNE

2. (6 pts) Draw a contour map of the function \( f(x, y) = x^3 - y \) showing level curves for the values \( k = -1, k = 0, \) and \( k = 1. \)

   \[
   \begin{align*}
   k = -1 & \quad -1 = x - y \\
   & \quad y = x + 1 \\
   k = 0 & \quad 0 = x^3 - y \\
   & \quad y = x \\
   k = 1 & \quad 1 = x^3 - y \\
   & \quad y = x - 1
   \end{align*}
   \]

3. (6 pts) Sketch the domain of the function \( f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}. \)

   \[
   \begin{align*}
   y & \geq 0 \text{ for } y \\
   x^2 + y^2 & \leq 25 \\
   & \text{ for } \sqrt{25 - x^2 - y^2}
   \end{align*}
   \]
4. Consider the surface defined by the function \( f(x, y) = ye^{x+y} \).

(a) (5 pts) Find the equation of the tangent plane to the surface at the point where \( x = 0 \) and \( y = 2 \).

Note \( f(0,2) = 2 \)

Let \( F(x, y, z) = 0 \) be defined by \( z = ye^{x+y} \rightarrow z - ye^{x+y} = 0 \)

\( \nabla F \) is orthogonal to \( F(x, y, z) \)

\( \nabla F = \langle -y^2e^{xy}, e^{xy} - xy e^{xy}, 1 \rangle \)

\( \nabla F(0,2,2) = \langle -4, -1, 1 \rangle \)

\[-4(x-0) - 1(y-2) + 1(z-2) = 0\]

or

\[-4x - y + z = 0\]

or

\[4x + y - z = 0\]

(b) (5 pts) Find the directional derivative of \( f \) in the direction of \( \mathbf{v} = (-1, 3) \) at the point where \( x = 0 \) and \( y = 2 \).

\[|\mathbf{v}| = \sqrt{10}\]

\[\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{10}} (-1, 3) = \mathbf{u}\]

\[\nabla f = \langle ye^{xy}, e^{xy} + ye^{xy} \rangle\]

\[\nabla f(0,2) = \langle 4, 1 \rangle\]

\[D_{\mathbf{u}} f(0,2) = \nabla f \cdot \mathbf{u} = \langle 4, 1 \rangle \cdot \langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle = -\frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}}\]

\[D_{\mathbf{u}} f(0,2) = -\frac{1}{\sqrt{10}}\]

(c) (5 pts) If a person is standing on the surface where \( x = 0 \) and \( y = 2 \) and she wants to travel downhill at the fastest rate, in which direction should she walk? What angle does this make with the positive \( x \)-axis?

Downhill

\(-\nabla f \text{ so in the direction } \langle -4, -1 \rangle\)

\[\tan \theta = \frac{1}{4}\]

\[\theta \approx 19.4^\circ\]

\[\approx \theta \approx 3.39^\circ\]
5. Consider the function \( T = \frac{v}{2u + v} \) where \( u = pq\sqrt{r} \) and \( v = pr\sqrt{q} \).

(a) (6 pts) Find \( \frac{\partial T}{\partial q} \). Do not simplify your answer.

\[
\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial T}{\partial v} \frac{\partial v}{\partial q} = \frac{2u}{(2uv)^2} + \frac{2u}{(2uv)^2} = \frac{2u}{(2uv)^2}
\]

\[
\frac{\partial T}{\partial u} = \frac{-2v}{(2uv)^2}, \quad \frac{\partial T}{\partial v} = \frac{2u}{(2uv)^2}
\]

\[
\frac{\partial u}{\partial q} = p\sqrt{r}, \quad \frac{\partial v}{\partial q} = \frac{p\sqrt{r}}{2\sqrt{q}}
\]

(b) (6 pts) Evaluate your answer from part (a) for \( p = 2, \ q = 1, \) and \( r = 4 \).

\[
\frac{\partial T}{\partial q} \bigg|_{p=2, \ q=1, \ r=4} = \frac{-2(2)}{(8+1)^2} (2\sqrt{4}) + \frac{2(4)}{(8+1)^2} \left( \frac{1.44}{2.56} \right) = \frac{-16}{256} + \frac{8}{256} = \frac{-32}{256} = \frac{-1}{8}
\]

6. (6 pts) For a function \( G(u, v, w) \) where \( u = u(p, q, r, s) \), \( v = v(p, q, r, s) \), and \( w = w(p, q, r, s) \), use the chain rule to write an expression that represents the partial derivative of \( G \) with respect to \( r \). That is, state \( \frac{\partial G}{\partial r} \).
7. (6 pts) For the surface defined by the equation $4xy - yz^3 = x^2z^4 + 10$:

(a) (6 pts) Find the normal line (the line orthogonal to the surface) at the point $(2, 2, 1)$.

$$4x - y - 3yz^2 - 12 = 0$$

$$\nabla F = \langle 4y - 2x - 3yz^2, 4x - y - 12y^2, -3yz^2 - 4xz^3 \rangle$$

$$\nabla F(2, 2, 1) = \langle 4, 7, -22 \rangle$$

$$x = 2 + 4t$$
$$y = 2 + 7t$$
$$z = 1 - 22t$$

(b) (6 pts) Find $\frac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$F_x = 4y - 2x - 3yz^2$$

$$F_z = -3yz^2 - 4xz^3$$

So

$$\frac{\partial z}{\partial x} = -\frac{4y - 2x - 3yz^2}{-3yz^2 - 4xz^3}$$

8. (6 pts) Use differentials to estimate the amount of aluminum used to make a closed box with length 10 cm, width 12 cm, and height 5 cm if the aluminum is 0.04 cm thick.

$$V = xyz$$

$$x = 10, y = 12, z = 5$$

$$dx = dy = dz = 0.04$$

$$dV = V_x dx + V_y dy + V_z dz$$

$$dV = \gamma z dx + xz dy + xy dz$$

$$dV = (10)(5)(0.04) + (10)(5)(0.04) + (10)(12)(0.04)$$

$$dV = 18.24$$

Approximately 18.24 cm$^3$ of Aluminum
9. (10 pts) Find all the critical points of the function \( f(x,y) = x^4 + 2y^2 - 4xy \). Then use the second derivative test to determine whether each is a local maximum, local minimum, or a saddle point.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x^3 - 4y \\
\frac{\partial f}{\partial y} &= 4y - 4x
\end{align*}
\]

Critical Points

\((0,0)\) \quad \((-1,-1)\) \quad \((1,1)\)

\[
D = \left[ \frac{\partial^2 f}{\partial x^2} \right] \left[ \frac{\partial^2 f}{\partial y^2} \right] - \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^2
\]

\[
D = 48x^2 - 16
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{C. P.} & D & \frac{\partial^2 f}{\partial x^2} \\
\hline
(0,0) & -16 & 12 \\
(-1,-1) & 32 & 12 \\
(1,1) & 32 & 12 \\
\hline
\end{array}
\]

\((0,0)\) is a saddle
\((-1,-1)\) is a local min
\((1,1)\) is a local min
10. (10 pts) Use Lagrange multipliers to find the maximum and minimum values (if they exist) of the function \( f(x,y) = xy + x + y \) subject to the constraint \( xy = 4 \)

\[
\nabla f = \langle y+1, x+1 \rangle \quad \nabla g = \langle y, x \rangle
\]

\( \nabla f = \lambda \nabla g \) gives

\[
\begin{align*}
y + 1 &= \lambda y \\
x + 1 &= \lambda x
\end{align*}
\]

subtract

\[xy = 4\]

\[x - y = \lambda x - \lambda y\]

\[(x - y) - (\lambda x - \lambda y) = 0\]

\[(x - y)(1 - \lambda) = 0\]

\[x = y \quad \text{or} \quad \lambda = 1\]

If \( x = 1 \)

\[y + 1 = y \quad \text{a contradiction so} \quad x \neq 1\]

If \( x = y \) then

\[xy = 4 \quad \rightarrow \quad x^2 = 4 \quad \rightarrow \quad x = \pm 2\]

Points \((2, 2)\) or \((-2, -2)\)

\[f(2, 2) = 2(2)x2 + 2 = 8\]

\[f(-2, -2) = (-2)(-2) - 2 - 2 = 0\]

\[f(2, 2) = 8 \quad \max\]

\[f(-2, -2) = 0 \quad \min\]