Exponents, Parentheses, and the Order of Operations

I. Exponential Notation (Shorthand for repeated multiplication)
   A. Vocabulary
      1) Consider
         \[ 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \]
         This can be written as \( 7^5 \) where 7 is called the **base** and 5 is called the **exponent**.
         \( 7^5 \) is called the **fifth power of seven**, or **seven to the fifth power**, or **seven to the fifth**.
      2) When the exponent is a 2, such as \( m^2 \), we say “**m squared**”.
         When the exponent is a 3, such as \( m^3 \), we say “**m cubed**”.

   Example 1   Express in exponential form:
   a) \( 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \)       b) \( 2x \cdot 2x \cdot 2x \cdot 2x \cdot 2x \)

   Example 2   Evaluate:
   a) \( 9^2 \)       b) \( (-9)^3 \)       c) \( \left( \frac{3}{5} \right)^2 \)
   d) \( 8^1 \)       e) \( (-1)^5 \)       f) \( (-5x)^4 \)

   B. So, for any natural number \( n \),
   \( b^n \) means “\( b \cdot b \cdot b \cdot b \cdot b \cdots \)\)” so that there are “\( n \)” factors

   Can you tell the difference between
   \[ m + m + m + m + m + m + m + m \] and \[ m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \]?

   What is the exponent of each factor in the following expression? \( 6xy^2z^2 \)
II. \(-x^2\) vs \((-x)^2\)

A. \(-x^2\) read “the opposite of x squared” or “negative x squared” means \(-1 \cdot x \cdot x\) so only the x is squared. The exponent 2 only applies to the number or variable just preceding it.

B. \((-x)^2\) read “the opposite of x, quantity squared” or “negative x, quantity squared” means \((-x)(-x)\) or \((-1) \cdot x \cdot (-1) \cdot x\) or \((-1)(-1)(x)(x)\). The exponent 2 applies to the entire parentheses just preceding it.

Example 3 Evaluate.

a) \(-10^2\)  
b) \((-10)^2\)  
c) \(-4^2\)  
d) \((-4)^2\)  
e) \(-2^4\)  
f) \((-2)^4\)

III. Order of Operations

A. In what order do we do arithmetic? If we didn’t have these rules, we would all get different answers to the same arithmetic problem.

Here is the Order of Operations:

1) Do all operations above and below any fraction bar, following the steps below.
2) Do operations inside grouping symbols (including absolute value bars), following the steps below. Work from the innermost grouping symbol to the outermost grouping symbol. The grouping symbols are:
   - parentheses \((\ )\)
   - brackets \([\ ]\)
   - braces \(\{\}\)
   - the fraction bar
   - absolute value bars \(|\ |
3) Simplify exponential expressions.
4) Do multiplications or divisions as they occur LEFT TO RIGHT.
5) Do additions or subtractions as they occur LEFT TO RIGHT.

B. When you see a problem like \((7 + 3(4 + 7(2 – 8(5 – 9))))\) we say that we have nested parentheses and we work from the innermost parentheses to the outermost parentheses.

C. Parentheses, brackets, or braces may be used
   1) to change the order of operations to be followed, as \(8 + 2 \cdot 5\) vs \((8 + 2) \cdot 5\), or
   2) to help clarify the problem, as \(3 \cdot 8 + 2 \cdot 5\) vs \((3 \cdot 8) + (2 \cdot 5)\)
Example 4  Using the *Order of Operations*, evaluate the following:

a) \(3 - 4 \cdot 2\)  
b) \(32 - 8 \div 4 - 2\)

c) \((4 \cdot 5)^2\)  
d) \(4 \cdot 5^2\)

e) \(3 + 2 \cdot 4^2 - 8\)  
f) \((14 \div 2) + 5(3 - 2)^2\)

g) \(-5 + 4[-3 + (100 \div 5^2)]\)  
h) \(\frac{4}{7} - \frac{3}{5} \cdot \frac{2}{9}\)

i) \(\frac{16 + |5 - 13| + 4^2}{17 - 5}\)  
j) \(-9 - 72 \div 8 \cdot 3^2 + 5\)
Example 5  Write the following statements as mathematical expressions using parentheses and brackets and then evaluate.

Multiply 9 by 6. Add 7 to this product. Subtract 12 from this sum. Divide this difference by 5.

IV. Evaluating Expressions Containing Variables

Example 6  Evaluate the following expressions for the given value of the variable.

a) $8x - 7$ when $x = 4$  
b) $x^2$ and $-x^2$ when $x = 9$  
c) $y^2$ and $-y^2$ when $y = -6$  
d) $2x^2 - 3x + 4$ when $x = \frac{2}{3}$  
e) $-y^2 + 2(x + 7) - 3$ when $x = -2$ and $y = -1$