Math 5C -- Calculus III **FINAL EXAM**
M. Eastman -- Spring 1998

1. Given the limaçon $r = 3 + 2sin\,\theta$:

   a. Make a careful sketch of this curve.

   b. Find the equation (in rectangular coordinates) of the line tangent to this curve at the point where it crosses the line $\theta = 0$.

   c. Find the area of the region inside this curve that is outside of the circle $r = 4$.

2. Given the points $(\,1,\,0,\,1\,)$, $(\,2,\,4,\,0\,)$, and $(\,0,\,0,\,3\,)$:

   a. Find the area of the triangle formed by these points.

   b. Find the equation of the plane determined by these points.

   c. Find the angle that the (upwardly oriented) normal vector to this plane makes with each positive coordinate axis.

3. Given the curve defined by $\boldsymbol{r}(t) = \;(ln(t^2+1))\boldsymbol{i} + \left(\sqrt{t^2+1}\right)\boldsymbol{j} + (tan^{-1}t)\boldsymbol{k}$

   a. Find the vector equation of the line tangent to this curve when $t = 1$.

   b. Set up the integral that calculates the length of this curve between $t = 0$ and $t = 1$. Do *not* evaluate the integral.

4. Given $f(x, y, z) = x^3y\; + 2y^2z + 5xz^2$, find:

   a. $f_{zy}(x, y, z)$

   b. $\nabla f(\,1,\,-2,\,3\,)$

   c. div$(\nabla f(x, y, z))$

   d. the directional derivative of $f(x, y, z)$ at $(\,1,\,-2,\,3\,)$ in the direction of the point $(\,0,\,0,\,1\,)$.

5. Given $f(x, y) = x^3 + y^2 - 3x - 6y$

   a. Find all the critical points of f.

   b. Find the function value of f at each critical point.

   c. Classify each function value as either a maximum, minimum, or saddle point.

6. Find the maximum and minimum values, and the points where each occurs, of $f(x, y, z) = 4x - 2y + z$ on the ellipsoid $4x^2 + y^2 + z^2 = 36$.

7. Evaluate by reversing the order of integration :

$$\int_0^2 \int_{x^2}^4 x \sqrt[3]{1 + y^2}\, dy\, dx$$

8. Given $\oint_{\mathbb{C}} (xy^2)dx + (2x^2y)dy$, where $\mathbb{C}$ consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$, followed by the line segment from $(2, 4)$ to $(0, 0)$, verify Green's theorem by:

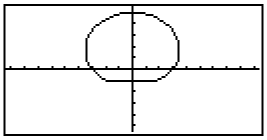
a. evaluating the line integral directly, and

b. evaluating the appropriate double integral.

9. Evaluate the surface integral $\int\int_{\mathbb{S}} (x^2z + y^2z)\, dS$, where $\mathbb{S}$ is the hemisphere $x^2 + y^2 + z^2 = 4$, and $z \geq 0$ [Hint: use cylindrical coordinates].

10. Use the Divergence Theorem to evaluate $\int\int_{\mathbb{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$, where $(x, y, z) = \left(ye^{z^2}\right)\boldsymbol{i} + (y^2)\boldsymbol{j} + (e^{xy})\boldsymbol{k}$, and $\mathbb{S}$ is the surface of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = y - 3$.

1. a.



b. $y = \frac{3}{2}x - \frac{9}{2}$

c. $\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[(3 + 2\,sin\,\theta)^2 - 4^2\right] dx = \ldots = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3}$

2. a. 4.5 square units

b. $8x - y + 4z = 12$

c. w/ x-axis: $cos^{-1}\left(\frac{8}{9}\right)$ w/ y-axis: $cos^{-1}\left(-\frac{1}{9}\right)$ w/ z-axis: $cos^{-1}\left(\frac{4}{9}\right)$

3. a. $< ln\,2 + t,\ \sqrt{2} + \frac{t}{\sqrt{2}},\ \frac{\pi}{4} + \frac{t}{2} >$

b. $\int_0^1 \frac{\sqrt{t^4+5t^2+1}}{t^2+1}\,dt$

4. a. $4y$

b. $<\ 39,\ -23,\ 38\ >$

c. $6xy + 4z + 10x$

d. $-\frac{161}{3}$

5. a. $(\,1,\,3\,)$, $(\,-1,\,3\,)$

b/c. $f\,(1,3) = -11$ (minimum) $f(-1,3) = -7$ (saddle)

6. $f_{max} = 18$ @ $(\,2,\,-4,\,2\,)$ $f_{min} = -18$ @ $(\,-2,\,4,\,-2\,)$

7. $$\int_0^4 \int_0^{\sqrt{y}} x\sqrt[3]{1+y^2}\,dx\,dy = \ldots = \frac{3}{16}\left(17\sqrt[3]{17}-1\right)$$

8. $$\int_{\mathbb{C}_1(y=x^2)} P\,dx + Q\,dy \;+ \int_{\mathbb{C}_1(y=2x)} P\,dx + Q\,dy$$

$$= \int_0^2 5x^5\,dx \;+ \int_2^0 12x^3\,dx = \frac{160}{3} - 48 = \frac{16}{3}$$

9. $$\int_0^{2\pi}\int_0^2 r^2\left(\sqrt{4-r^2}\right)\left(\sqrt{\tfrac{r^2}{4-r^2}+1}\right)(r\,dr\,d\theta) = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = 16\pi$$

10. $$\int\int\int_{\mathbb{E}} (2y)\,dV = \int_0^{2\pi}\int_0^3\int_{r\,sin\,\theta-3}^0 (2\,r\,sin\,\theta)\,r\,dz\,dr\,d\theta = -\,\frac{81\pi}{2}$$