Math 19B -- Calculus II

M. Eastman -- Winter 2003 **Midterm 2 - Chapters 7, 8, 11**

1. Evaluate:

   a. $\int \frac{3x-4}{x^3+x}\,dx$ b. $\int x^2 sin\,x\,dx$ c. $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

2. The region in the first quadrant bounded by $y = ln\,x$ and $x = e^2$ is rotated around the y-axis. Find the volume of the solid that is generated.

3. Determine whether each integral is convergent or divergent. Justifiy your answer. Evaluate any integral that is convergent.

   a. $\int_1^\infty \frac{1}{x^2+1}\,dx$ b. $\int_0^4 \frac{1}{x\sqrt{x}}\,dx$

4. Set up the definite integral that would find the length of the part of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that lies in the first quadrant. *Do not evaluate the integral*.

5. The part of the curve $y = x^3$ from $(0,0)$ to $(1,1)$ is rotated around the x-axis. Find the area of the surface that is generated.

6. Determine whether the sequences converge or diverge. Justify your answer. If a sequence converges, find the limit.

   a. $a_n = \frac{\sqrt{n}}{\sqrt[3]{n}+\sqrt[4]{n}}$ b. $\left\{ \frac{ln(2\cdot 3^n)}{5\,n} \right\}_{n=1}^{\infty}$

7. Determine whether the series converges or diverges. Justify your answer. If the series converges, find its sum.

   a. $\sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+4} \right)$ b. $\sum_{n=1}^{\infty} \frac{3^n+(-4)^n}{6^n}$

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1. a. Use partial fractions: $\int \frac{-4}{x} + \frac{4x+3}{x^2+1}\,dx = -4ln|x| + 2ln|x^2+1| + 3tan^{-1}x + C$

   b. Use integration by parts (twice): $\int x^2 sin\,x\,dx = -x^2 cos\,x + 2x\,sin\,x + 2\,cos\,x + C$

   c. Use trig substitution: $sin\,\theta = \frac{x}{2}$ $\int_0^{\pi/6} sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/6}(1 - cos2\theta)\,d\theta = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

2. Using shells: $\int_0^{e^2}(2\pi x)(ln\,x)dx$ Using washers: $\int_0^2 \pi(e^y)^2 dy$ Volume is $\frac{(3e^4+1)\pi}{2}$

3. a. $\lim_{M\to\infty}\int_1^M \frac{1}{x^2+1}\,dx = \lim_{M\to\infty}(tan^{-1}x)\Big|_1^M = \lim_{M\to\infty}(tan^{-1}M - tan^{-1}1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

   b. $\lim_{a\to 0^+}\int_a^4 \frac{1}{x\sqrt{x}}\,dx = \lim_{a\to 0^+}\frac{-2}{\sqrt{x}}\Big|_a^4 = \lim_{a\to 0^+}\left(\frac{-2}{\sqrt{(4)}} - \left(\frac{-2}{\sqrt{(a)}}\right)\right)$

   Since $\lim_{a\to 0^+}\left(\frac{-2}{\sqrt{(4)}} - \left(\frac{-2}{\sqrt{(a)}}\right)\right)$ does not exist, the integral diverges.

4. Solve for a variable (either method I or II below is sufficient):

I. $y = \sqrt{9\left(1 - \frac{x^2}{4}\right)} = \frac{3}{2}\sqrt{4 - x^2}$ $\frac{dy}{dx} = \frac{3}{2}\left(\frac{1}{2}\frac{-2x}{\sqrt{4-x^2}}\right) = \frac{-3x}{2\sqrt{4-x^2}}$

$L = \int_0^2 \sqrt{1 + \left(\frac{9x^2}{16-4x^2}\right)}\,dx$

II. $x = \sqrt{4\left(1 - \frac{y^2}{9}\right)} = \frac{2}{3}\sqrt{9 - y^2}$ $\frac{dx}{dy} = \frac{2}{3}\left(\frac{1}{2}\frac{-2y}{\sqrt{9-y^2}}\right) = \frac{-2x}{3\sqrt{9-y^2}}$

$L = \int_0^3 \sqrt{\left(\frac{4y^2}{81-9y^2}\right) + 1}\,dy$

5. Area $= \int_0^1 2\pi(x^3)\sqrt{1+(3x^2)^2}\,dx = \int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx$

Let $u = 1+9x^4 \quad du = 36x^3\,dx$

Integral becomes $\int_1^{10} 2\pi\sqrt{u}\left(\frac{du}{36}\right) = \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du = \frac{\pi}{18}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{10}$

Surface Area is: $\frac{\pi}{27}\left(10\sqrt{10}-1\right)$

6. a. $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt[3]{n}+\sqrt[4]{n}}\left(\frac{1/\sqrt{n}}{1/\sqrt{n}}\right)\right) = \lim_{n\to\infty}\left(\frac{1}{n^{-1/6}+n^{-1/4}}\right) = \frac{1}{0+0}$ This diverges.

b. $\lim_{n\to\infty}\left(\frac{ln(2\cdot 3^n)}{5\,n}\right) = \lim_{n\to\infty}\left(\frac{ln2+n\cdot ln3}{5\,n}\left(\frac{1/n}{1/n}\right)\right) = \lim_{n\to\infty}\left(\frac{\frac{ln2}{n}+ln3}{5}\right) = \frac{0+ln\,3}{5}$

Conveges to $\frac{ln\,3}{5}$

7. a. This is a telescoping series: $\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\ldots$

$S_n = \frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4} \longrightarrow \frac{1}{3}+\frac{1}{4}-0-0 = \frac{7}{12}$

The series converges to $\frac{7}{12}$.

b. $\sum_{n=1}^{\infty}\frac{3^n+(-4)^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{3}{6}\right)^n+\left(\frac{-4}{6}\right)^n = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n$

Each of these is a geometric series with $|r|<1$, so each part converges

The sum converges to $\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)}+\frac{\left(-\frac{2}{3}\right)}{1-\left(-\frac{2}{3}\right)} = 1-\frac{2}{5} = \frac{3}{5}$