Chapter 3: Vectors

Overview of the chapter:

- graphical representation of vectors using the parallelogram rule
- vector component method for adding/subtracting vectors

Vectors have length and direction that are given by the right triangle relationships that are very familiar with everybody:

\[ r = \sqrt{r_x^2 + r_y^2} \]

\[ \tan \theta = \frac{r_y}{r_x}, \quad \cos \theta = \frac{r_x}{r}, \quad \sin \theta = \frac{r_y}{r} \]

The dimension of the space defines the type of “animal” it is.

- 1-dimensional objects are called **scalars** (only magnitude or number)
- 2,3-dimensional objects are called **vectors** (magnitude & direction)
- 4-dimensional objects are manifolds (hypercube)
- 11 or 26-dimensional objects maybe “branes”
- ∞-dimensional objects in QM live in Hilbert space

**GRAPHICAL PROPERTIES OF VECTORS**

A vector has a head and a tail:

In order for two vectors to be identical, they must have the same length and direction. Vectors with the same magnitude but different directions are not the same vectors (their angles different by 180°). For example,

Vector magnitudes have physical significance when attached to physical quantities. For example, in a barroom brawl with Mike Tyson it is important to be able to throw a punch with good magnitude and correct direction. What does that mean? Mike Tyson's punch compared to Carlos' might look something like

It instantly gives a physical significance to the differences in punch strength.

When adding/subtracting parallel (or antiparallel) vectors, the result or **resultant** of these combined vectors is as simple as adding them like adding/subtracting scalars. A straightforward example is riding your bike in the wind, if riding with or against the wind, the difference is obvious as shown by the vector addition:

However, most vectors are NOT parallel (or antiparallel) and therefore, require a different technique to adding/subtracting vectors. This technique is called the **Parallelogram Rule**. Let’s add to vectors \( \vec{r}_1 + \vec{r}_2 = \vec{R} \) (≡ resultant) using this rule:
When subtracting two vectors, 

Adding more than two vectors is a bit more complicated but not much more. Again, one uses the parallelogram rule. Let’s add 3 vectors two different ways ( \((A+B)+C\) or \(A+(B+C)\) ) that end in with the same result.

**VECTOR COMPONENT METHOD**

When dealing with “one-dimensional vectors” it was fairly straightforward to add them because they added like scalars and this helped us to visualize motion problems when an object was speeding up or slowing down. These were graphical descriptions of one-dimensional vectors. As we move onto more complicated situations like two-dimensional motion, graphical methods are more cumbersome to extract numbers but will be immensely helpful in visualizing what is going on physically. What we need is a quicker and more accurate way to determine numerical results when vectors are added or subtracted. This is achieved by using **vector components to find resultants using the parallelogram rule**.

Coordinate systems are broken down into four quadrants and each quadrant has a designated sign value for the \(x\)- and \(y\)-components of a vector. Knowing these well will be very useful when breaking vectors into components. All vectors that are added or subtracted using the Vector Component method, have the vectors written in terms of unit vectors. Unit vectors indicate a specific direction along one of the coordinate axes. Just like everyone of you are very special, unit-vectors are also very special because their lengths (magnitudes) are exactly one.

When a vectors is broken down into components, a vector can be written as “walking horizontally” along the \(x\)-axis a distance of \(a_x\) and then “walking vertically” along the \(y\)-axis a distance of \(a_y\) as shown on the right. However, it is more common to project the vector onto the axes to obtain the \(a_x\) and \(a_y\) values. To write this mathematically, we start with two vectors \(a\) and \(b\)

\[
\vec{a} = a_x \hat{i} + a_y \hat{j} \quad \text{and} \quad \vec{b} = b_x \hat{i} + b_y \hat{j}
\]
and add them via components:

\[ \vec{R} = \vec{a} + \vec{b} = (a_x \hat{i} + a_y \hat{j}) + (b_x \hat{i} + b_y \hat{j}) = (a_x + b_x, a_y + b_y) \]

To then determine the magnitude and direction (angle) of \( \vec{R} \), we apply the usual magnitude and direction relationships:

\[
\begin{align*}
|\vec{R}| &= \sqrt{\vec{R}_x^2 + \vec{R}_y^2} = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \\
\theta &= \tan^{-1}(\frac{\vec{R}_y}{\vec{R}_x}) = \tan^{-1}(\frac{a_y + b_y}{a_x + b_x})
\end{align*}
\]

Graphically and mathematically, what we see when we add two different vectors together looks like this:

**Example 3.1**

A person walks in the following pattern: 3.1 km north, then 2.4 km west, and finally 5.2 km south. (a) Sketch the vector diagram that represents this motion. (b) How far and (c) in what direction would a bird fly in a straight line from the same starting point to the same final point?

**Solution**

a. We first setup our coordinate system in the standard way where the direction east lies along the +x-direction and north lies along the +y-direction. If we label the individual displacement vectors \( 
\vec{A}, \vec{B}, \text{ and } \vec{C} \)
and the total displacement as \( \vec{D} \). The vector diagram representing the motion looks like the one shown on the right.

b. The final point is represented by the total displacement vector and is given by

\[ \vec{D} = \vec{A} + \vec{B} + \vec{C} = (A_x + B_x + C_x, A_y + B_y + C_y) \]

\[ = (0 - 2.4 + 0, 3.1 - 5.2) = (-2.4, -2.1) = -2.4 \hat{i} - 2.1 \hat{j} = (D_x, D_y) \]

To determine the magnitude of the total displacement, we use the length formula:

\[ |\vec{D}| = D = \sqrt{D_x^2 + D_y^2} = \sqrt{(-2.4)^2 + (-2.1)^2} \approx 3.2 \text{ km} = D. \]

The angle formula gives us

\[ \theta = \tan^{-1}(\frac{D_y}{D_x}) = \tan^{-1}(\frac{-2.1}{-2.4}) = 41^\circ \]

This implies that the total displacement lies in the first quadrant, however, we know physically that it must lie in the third quadrant. When this is the case, we need to add 180° to get the correct angle (as well as into the correct quadrant):

\[ \theta = 180^\circ + 41^\circ = 221^\circ = \theta \]

One can see why it is always important to sketch the vector diagram beforehand to see where that vector must lie instead of blindly trusting the mathematics.
**Example 3.2**

A hiker undergoes three successive displacements as follows: 4.00 km southwest, then 5.00 km east, and finally 6.00 km in a direction 60° north of east. (a) Sketch the vector diagram that represents this motion. (b) What is the direction and magnitude of the hiker’s net displacement?

**Solution**

a. We first setup our coordinate system in the standard way where the direction east lies along the +x-direction and north lies along the +y-direction. If we label the individual displacement vectors \( \mathbf{d}_1, \mathbf{d}_2, \) and \( \mathbf{d}_3 \) and the total displacement as \( \mathbf{D} \). There is two ways to draw the vector diagram representing this motion. The first vector diagram is the actual path that the hiker took. However, calculational wise, this vector diagram is cumbersome to use. The “correct” calculational diagram is taking all of the vectors and placing their tails at the origin, which is considerably easier to use.

b. By using the calculational diagram on the right, we can just pick-off the x- and y-components from the diagram.

<table>
<thead>
<tr>
<th></th>
<th>x-component</th>
<th>y-component</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{d}_1 )</td>
<td>( d_1 \cos \theta_1 = -4 \cos 45 = -2.83 )</td>
<td>( d_1 \sin \theta_1 = -4 \sin 45 = -2.83 )</td>
</tr>
<tr>
<td>( \mathbf{d}_2 )</td>
<td>5 m</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbf{d}_3 )</td>
<td>6 cos 60 = 3 m</td>
<td>6 sin 60 = 5.20 m</td>
</tr>
</tbody>
</table>

To determine the magnitude and direction of the hiker’s net displacement, we use the length and angle formulas.

\[
D = \sqrt{D_x^2 + D_y^2} = \sqrt{(d_{1x} + d_{2x} + d_{3x})^2 + (d_{1y} + d_{2y} + d_{3y})^2}
\]

\[
= \sqrt{(-2.83 + 5.00 + 3.00)^2 + (-2.83 + 0 + 5.20)^2} = \sqrt{(5.17)^2 + (2.37)^2} = 5.69 \text{ m} = D
\]

\[
\theta = \tan^{-1} \left( \frac{D_y}{D_x} \right) = \tan^{-1} \left( \frac{5.17}{2.37} \right) = 24.6^\circ = \theta
\]

**Example 3.3**

An explorer is caught in a whiteout (in which the snowfall is so thick that the ground cannot be distinguished from the sky) while returning to base camp. He was supposed to travel due north for 5.6 km, but when the snow clears he discovers that he actually traveled 7.8 km at 50° north of due east. (a) Sketch the vector diagram that represents this motion. (b) How far and in what direction must he now travel to reach base camp?

**Solution**

a. The goal was to move due north along the vector \( \mathbf{D} \), but the whiteout caused the explorer to move along the vector \( \mathbf{d}_1 \). To reach the base camp, the explorer must move along path \( \mathbf{d}_2 \) such that

\[
\mathbf{D} = \mathbf{d}_1 + \mathbf{d}_2 \quad \rightarrow \quad \mathbf{d}_2 = \mathbf{D} - \mathbf{d}_1
\]
b. To determine the magnitude and direction of \( \mathbf{d}_2 \), we setup the components of \( \mathbf{D} \) and \( \mathbf{d}_1 \):

<table>
<thead>
<tr>
<th></th>
<th>x-component</th>
<th>y-component</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{D} )</td>
<td>0</td>
<td>5.6 km</td>
</tr>
<tr>
<td>( \mathbf{d}_1 )</td>
<td>7.8 ( \cos 50^\circ ) = 5.01 km</td>
<td>7.8 ( \sin 50^\circ ) = 5.98 km</td>
</tr>
</tbody>
</table>

To find the magnitude and direction \( \mathbf{C} \) (subtract the column values), we compute

\[
\mathbf{C} = \mathbf{A} - \mathbf{B} = (A_x - B_x, A_y - B_y) \\
= (0 - 5.01, 5.6 - 5.98) = (-5.01, -0.38) \text{ km}
\]

The magnitude and direction of the vector \( \mathbf{C} \) is

\[
d_2 = \sqrt{(D_x - d_{2,x})^2 + (D_y - d_{2,y})^2} = \sqrt{(0 - 5.01)^2 + (5.6 - 5.98)^2} \\
= \sqrt{(-5.01)^2 + (-0.38)^2} = 5.0 \text{ km} = d_1
\]

and the direction is

\[
\theta = \tan^{-1} \left( \frac{d_{2,y}}{d_{2,x}} \right) = \tan^{-1} \left( \frac{-0.38}{-5.01} \right) + 180^\circ = 4.3^\circ + 180^\circ = 184^\circ = \theta
\]

Now, let's go back and adjust our vector diagram because these calculations show that our sketch is not correct. The y-component \( d_{1y} \) is larger than the total displacement \( \mathbf{D} \); in other words, \( d_{1y} \) "over shot" the traverse and had to back track to basecamp.