\[
\int \frac{xe^x}{1 + e^x} \, dx
\]

ok, IBP doesn't work.
let's try a "weird" substitution

\[
\text{weird sub: my life might be easy if I let } w = \frac{1}{1 + e^x}
\]

\[
dw = \frac{1}{2(1 + e^x)^{\frac{3}{2}}} e^x \, dx
\]
and
\[
w^2 = 1 + e^x
\]
\[
w^2 - 1 = e^x
\]
so
\[
\ln(w^2 - 1) = x
\]

Try to rewrite the integral
\[
\int_0^{\ln(w^2 - 1)} 2dw = 2 \int \ln(w^2 - 1) \, dw.
\]

Yikes! now I can do IBP, no?

\[
\text{IBP}
\]

\[
u = \ln(w^2 - 1) \quad dv = dw
\]
\[
\frac{du}{w^2} = \frac{1}{2w^4}, 2wdw \quad v = w
\]
\[
= 2 \left[ w \ln(w^2 - 1) - \frac{\ln(w^2 - 1)}{w^2} \right]
\]

For \( \frac{2w^2}{w^2 - 1} \), we do by long division
\[
\frac{2w^2}{w^2 - 1} = 2 + \frac{2}{w^2 - 1}
\]

\[
\frac{2w^2}{w^2 - 1} = 2 + \frac{2}{w^2 - 1}
\]

\[
\text{partial fractions}
\]
\[
= 2 \left[ w \ln(w^2 - 1) - \int 2dw - \int \frac{2}{w^2 - 1} \, dw \right]
\]
\[
= 2 \left[ w \ln(w^2 - 1) - 2w - \int \frac{1}{w - 1} \, dw - \int \frac{1}{w + 1} \, dw \right]
\]
I think I'll be done when I do the P.F. bit

\[
\frac{2}{(w+1)(w-1)} = \frac{A}{w+1} + \frac{B}{w-1} \quad A = \frac{2}{2} = 1 \quad B = \frac{2}{2} = 1
\]

so

\[
\int \frac{xe^x}{1+e^x} \, dx = 2 \int \ln(w^2-1) \, dw
\]

\[
= 2 \left[ w \ln(w^2-1) - 2w + \int \frac{dw}{w+1} - \int \frac{1}{w-1} \, dw \right]
\]

\[
= 2w \ln(w^2-1) - 4w + 2 \ln(w+1) - 2 \ln(w-1) \quad \text{yikes.}
\]

Now, back to x...

\[
w = \frac{1}{1+e^x}
\]

\[
= 2 \ln \left( \frac{1+e^x}{1+e^x} \right) - 4 \ln(1+e^x) + 2 \ln \left| \frac{1+e^x + 1}{1+e^x - 1} \right| + C
\]

Phew!
\[ \int \frac{\sqrt{x} \, dx}{1 + x^3} \]

This is definitely going to be weird.

I have to think about what we have to do to get the \( \sqrt{x} \) piece as part of the "dw" piece your weird substitution.

Try \( W = x^{\frac{3}{2}} \)

Then \( dw = \frac{3}{2} x^{\frac{1}{2}} \, dx = \frac{3}{2} \sqrt{x} \, dx \)

so I'll have \( \sqrt{x} \, dx = \frac{2}{3} \, dw \ldots \)

\[ W = x^{\frac{3}{2}} \]

\[ x^3 = (x^{\frac{3}{2}})^2 = W^2 \]

Now rewrite the integral:

\[ \int \frac{\frac{3}{2} \, dw}{1 + W^2} = \frac{2}{3} \int \frac{dw}{1 + w^2} \]

Wow! \( \arctan \)!

\[ = \frac{2}{3} \arctan(W) \]

\[ = \frac{2}{3} \arctan((x^{\frac{3}{2}})) + C \]

So, whoever said \( \arctan \) was correct!
\[
\int \frac{\sin x \cos x}{\sin^3 x + \cos^4 x} \, dx = \int \frac{\sin x \cos x}{\sin^4 x + (1 - \sin^2 x)^2} \, dy
\]

Now, let \( u = \sin^2 x \)

(\text{yeah, I looked in The solutions book…})

\[ u = \sin^2 x \]
\[ du = 2 \sin x \cos x \, dx \]

The integral becomes

\[
\frac{1}{2} \int \frac{du}{u^2 + (1-u)^2}
\]

\[
= \frac{1}{2} \int \frac{du}{u^2 - 2u + 1} = \int \frac{du}{4u^2 - 4u + 2}
\]

\[
= \frac{1}{4} \int \frac{du}{(u-\frac{1}{2})^2 + 1}
\]

It will be arctangent here.

\[
\tan \Theta = 2u - 1 \quad \text{etc.}
\]