Chapter 3 - Probability

3.1 – Terminology

An *experiment* is a process that, when performed, results in exactly one of many observations. These observations are called the *outcomes* of the experiment. The set of all possible outcomes for an experiment is called a *sample space*.

ex. Suppose we roll a die once, and we get a 4.

Experiment:

Outcome:

Sample Space:

ex. Suppose we toss a coin twice, and we get a heads (H) on the first toss and a tails (T) on the second toss.

Experiment:

Outcome:

Sample Space:

A *Tree Diagram* shows the sample space for an experiment (and will later be very useful when finding probabilities, section 3.5)

ex. Draw a tree diagram for the experiment of tossing a coin three times. Then find the sample space of this experiment.
ex. Draw a tree diagram and find the sample space for the experiment of tossing one coin and then picking a marble out of a bag, which contains a red, yellow, and green marble.

An event is a collection of one or more of the outcomes of an experiment.

ex. Referring back to the experiment of tossing one coin three times, list the possible outcome(s) for the following events:

All three coins are heads =
All three coins are the same =
At most one of the coins is a tail =

Probability is a numerical measure of the likelihood that a specific event will occur. If A denotes an event, the probability for the event A is denoted P(A).

People often have some idea of what probability means, and they often think of it in percentages. In statistics we tend to not use percent notation, but we write probability in decimal form. For instance, many of you have an idea that there is a “fifty-fifty” chance of getting heads or tails when tossing a fair coin, implying that there is a 50% chance of getting a head, and a 50% chance of getting a tail. Or, if we tossed a coin many times, there would be about 50% heads and 50% tails. In statistics we would usually say and write this example as P(head) = 0.5 and P(tail) = 0.5.

One can also say that the probability of an event is the proportion of times the event occurs in the long run, as a probability experiment is repeated over and over again. This is often referred to as the law of large numbers.

The probability of an event always lie between ....... and ....... .

If P(A) = 1 , then
If P(A) is close to 1, then
If P(A) is close to 0, then
If P(A) = 0, then
A **probability model** consists of a sample space, along with a probability for each event. The sum of the probabilities of all individual events in a sample space is always 1.

ex. Show how the above property holds true for tossing a fair coin.

The following model is used to compute the probabilities of events for an experiment for which **all outcomes are equally likely**. That is, each event has the same probability of occurrence.

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}}$$

ex. Find the probability of obtaining a 4 when tossing a die.

ex. Find the probability of obtaining a number greater than 2 when tossing a die.

When the outcomes for an experiment are NOT equally likely, we often use the **Empirical Method**, which uses the proportion of times an outcome occurs (the relative frequency) as an approximation for the probability.

If an experiment is repeated $n$ times and an event $A$ is observed $f$ times, then, according to the Empirical Method,

$$P(A) = \frac{f}{n}$$

If we are considering a population, this will give us an exact probability. If we are considering a sample, this will give us an approximation, but if $n$ is large, the approximation will be close to the actual probability.

ex. A bag contains 4 yellow, 5 red, and 8 green marbles. If a marble is randomly selected from this bag, what is the probability of selecting a
   a) yellow marble       b) red marble       c) green marble
Out of the 3000 families who live in a given apartment complex in New York City, 600 paid no income tax last year. What is the probability that a randomly selected family from these 3000 families did pay income tax last year?

An unusual event is an event that is unlikely to happen. Unless otherwise stated in a problem, we will say that an event is unusual if its probability is less than 0.05.

ex. Suppose there are 35 winning lottery tickets in a total of 1000 tickets sold. Would it be considered unusual to buy a winning ticket if you bought one random ticket?

A compound event is an event that is formed by combining two or more events.

The event "A and B" is the event where A and B both occur.

The event "A or B" is the event where A occurs, or B occurs, or A and B both occur.

ex. Let $A=\{1, 2, 3, 4, 5\}$, $B=\{2, 4, 6, 8\}$

$A \text{ and } B =$
$A \text{ or } B =$

Conditional probability is the probability that an event will occur knowing (given) that another event has already occurred (or will occur simultaneously). If A and B are two events, then

\[ P(A|B) = \text{the probability of A given that B has already occurred} \]

A contingency table helps in determining probabilities, including conditional probabilities. If not provided in the table, start by summing up all rows and columns respectively. (section 3.4)
A contingency table of employees at a company

<table>
<thead>
<tr>
<th></th>
<th>18-29 yrs old</th>
<th>30-45 yrs old</th>
<th>46 yrs or older</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Male</strong></td>
<td>10</td>
<td>21</td>
<td>15</td>
</tr>
<tr>
<td><strong>Female</strong></td>
<td>15</td>
<td>18</td>
<td>7</td>
</tr>
</tbody>
</table>

If one person from this company was selected at random, find the following probabilities:

\[
P(\text{male}) =
\]

\[
P(\text{30-45 yrs old and Female}) =
\]

\[
P(\text{46 yrs or older}) =
\]

\[
P(\text{30-45 yrs old or Female}) =
\]

\[
P(\text{18-29 yrs old or 30-45 yrs old}) =
\]

\[
P(\text{18-29 yrs old | male}) =
\]

\[
P(\text{30-45 yrs old | female}) =
\]

\[
P(\text{female | 18-29 yrs old}) =
\]
3.2 – Independent and Mutually Exclusive Events

Events that cannot occur together are **mutually exclusive** events.

A **venn diagram** is a picture consisting of a box that represents the sample space S together with circles that represent events (section 3.5).

ex. Suppose we roll a die, and we are looking at the following events
A = a number less than 3 is observed
B = a number greater than 3 is observed
C = an even number is observed

Draw a venn diagram of events A and B. Are the events mutually exclusive?

Draw a venn diagram of events B and C. Are the events mutually exclusive?

Give other examples of mutually exclusive events:

If events A and B are mutually exclusive, then \( P(A \text{ and } B) = \) ............... 

The **complement** of an event A is the event that A does not occur, denoted \( A' \) or \( A' \). 

ex. If event A is to roll an even number with a die, then event \( A' \) is ...

ex. If event B is to pick a blue marble from a bag with mixed colored marbles, then \( B' \) is ...

\( P(A) + P(A') = \)

**The Rule of Complements:** \( P(A') = \)
ex. Based on the numbers in the contingency table in 3.1, find

\[ P(\text{less than 46 yrs old}) = \]

Two events A and B are **independent** if the occurrence of one does not affect the probability of the occurrence of the other. That is, A and B are independent if

either \( P(A|B) = P(A) \) or \( P(B|A) = P(B) \).

If A and B are **not** independent, they are said to be **dependent**.

**Examples of dependent events:**

ex. Referring back to the two way classification of employees, are the events “female (F)” and “18-29 yrs old (A)” independent events?

### 3.3 –Basic Rules of Probability

**The Multiplication Rule:**

\[ P(\text{A and B}) = P(B) \cdot P(\text{A|B}) \]

If two events A and B are independent, then

\[ P(\text{A and B}) = P(B) \cdot P(\text{A}) \]

**The General Addition Rule:**

\[ P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B}) \]

If A and B are mutually exclusive events, then

\[ P(\text{A or B}) = P(\text{A}) + P(\text{B}) \]

**The General Method for computing Conditional Probabilities:**

\[ P(A|B) = \frac{P(\text{A and B})}{P(B)} \quad \text{where } P(B) \neq 0 \]
ex. The probability that a person is allergic to a certain drug is 0.04. Suppose we pick three random people. Draw a tree diagram with probabilities. (section 3.5)

What is the probability that none of three randomly picked persons is allergic to this drug?

What is the probability that exactly one of three randomly picked persons is allergic to this drug?

**Sampling with replacement** means that we replace/return a selected item before sampling the next one, so the first item is available to be selected again.

**Sampling without replacement** means that we leave any previously selected items out when sampling the next one, so any one item can only be selected at most once.

ex. Construct a tree diagram with probabilities, for randomly picking 3 marbles out of a bag containing 5 red and 7 blue marbles. The marbles are picked **without replacement**.

Find the probability of picking all blue marbles.

Find the probability of picking 1 red and 2 blue marbles.
When sampling with replacement, the draws are ................................................. *(dependent or independent)*

When sampling without replacement, the draws are ........................................... *(dependent or independent)* unless the sample is very small compared to the population (less than 5%), in which case we can consider it ................................................. .

The probability that an event occurs **at least once** can often be found more easily using the probability of the **compliment** subtracted from 1: \( P(\text{At least one}) = 1 - P(\text{None}) \)

*ex.* *Find the probability of picking at least one red marble from the example above.*

**Additional Probability Topic – Counting** *(this topic is NOT in the textbook)*

For a sequence of two events in which the first event can occur \( m \) ways and the second event can occur \( n \) ways, the events together can occur a total of \( m \cdot n \) ways.

*ex.* *How many outcomes do we have for an experiment where we toss a coin and roll a die?*

*ex.* *How many outcomes do we have for an experiment where we toss a coin three times?*

*ex.* *Suppose you have 4 pair of pants, 10 shirts, 2 hats, and 3 pair of shoes. How many different outfits are possible with this selection, assuming you pick one piece from each category?*

*ex.* *A restaurant menu has 5 different appetizers, 9 different entrees, and 3 different desserts. If a customer orders one appetizer, one entree, and one dessert, how many different meals are possible?*
**Example:** A friend of mine once accidentally put my ATM card into the ATM and then entered her own code, and it worked! It turned out the bank had given us the same four digit code. Suppose the codes had been generated randomly, then what is the probability of my friend being given the same code as myself?

**Example:** Suppose you have to choose a password for your online banking account, consisting of 8 characters, which can be either letters or numbers. How many possible passwords are there?

**Permutations** means ordering. Counting number of permutations means we are counting the number of different ways that a group of items can be ordered.

**Example:** Suppose I remember that my four digit code has the numbers 0, 1, 7, 8 in it, but I can’t remember the order. How many different combinations of these four numbers are there?

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The factorial symbol \( ! \) denotes the product of decreasing positive whole numbers. For example, \( 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \).

By definition, \( 0! = 1 \).

5! is read "5 factorial".

The factorial symbol can be found on your calculator: Press \( \text{MATH} \), select PRB, then select the ! symbol.

**Example:**

\[
\begin{align*}
\text{Find:} & \quad 8! \\
& \quad (10 - 6)! \\
& \quad (12 - 12)!
\end{align*}
\]

The number of permutations of \( n \) objects is \( n! \).
The number of ways that we can select and order \( r \) objects out of \( n \) is
\[
_nP_r = \frac{n!}{(n-r)!} = n (n-1) ... (n-r+1)
\]

\( _nP_r \) is read as "the number of permutations (or arrangements) of \( r \) items from \( n \) items." Note that we are selecting \( r \) items without replacement and that the order matters.

You can calculate permutations on your calculator:
First enter the value of \( n \). Then press [MATH] select PRB, then select the \( _nP_r \) symbol. Last enter the value of \( r \) and press [ENTER].

ex. Suppose our class starts a Statistics Club here at Cabrillo, and we want to select one student from the class to be the president, another to be the secretary, and a third to be the treasurer. If we randomly select these three people, out of the total students, how many different arrangements of candidates are possible?

ex. An ice cream shop offers 30 flavors of ice-cream. Suppose you are going to buy a two scoop ice-cream cone, and you want two different flavors. If you do care how the ice-cream scoops are arranged, how many ways are there to select and arrange your 2 scoops?

A combination is a distinct group of selected object (a subset), in which order is irrelevant.

The number of ways that we can select \( r \) objects out of \( n \) is
\[
_nC_r = \frac{n!}{r!(n-r)!}
\]

\( _nC_r \) is read as "the number of combinations of \( r \) items chosen from \( n \)." Note that we are selecting \( r \) items without replacement and that the order does not matter.

You can calculate permutations on your calculator:
First enter the value of \( n \). Then press [MATH] select PRB, then select menu item 3 (which is the \( _nC_r \) symbol). Last enter the value of \( r \) and press [ENTER].

ex. Suppose our Statistics Club need to select a 3 person committee to investigate some peculiar probability problems. The three people are randomly selected out of the … club members. In how many ways can this committee be formed?
You are now ready for another 2 scoop ice-cream, still with 30 flavors to choose from. You would like two different kinds of flavor, but you have decided that the order doesn't matter to you. How many different ice-cream combinations do you have to choose from?

**Some more fun counting and probability problems**

1. Your friend wants a 2 scoop ice-cream too. The order matters to her, but she doesn't necessarily need to have two different flavors. How many ice cream combinations does she have to choose from?

2. On a math test there are 10 multiple choice questions with 4 possible answers each, and 15 true-false questions. In how many possible ways can the 25 questions be answered? If you randomly guess on all problems, what is the probability that you get all of them right?

3. In California's Super Lotto Plus lottery game, winning the jackpot requires that you select the correct five numbers between 1 and 47 inclusive and, in a separate drawing, you must also select the correct single number between 1 and 27 inclusive. Find the probability of winning the jackpot.

4. A box contains 10 red marbles and 10 green marbles.

   (a) Sampling at random from the box five times with replacement, you have drawn a red marble all five times. What is the probability of drawing a red marble the sixth time?

   (b) Sampling at random from the box five times without replacement, you have drawn a red marble all five times. Without replacing any of the marbles, what is the probability of drawing a red marble the sixth time?
(c) You have tossed a fair coin five times and have obtained heads all five times. A friend argues that a tail is due to occur and, hence, the probability of obtaining a head on the sixth toss is less than 0.50. Is he right? Is coin tossing mathematically equivalent to the procedure mentioned in part (a) or the procedure mentioned in part (b)? Explain.

5. A pizza parlor has 12 different toppings available for its pizzas, and 2 of these toppings are pepperoni and anchovies. If a customer picks 2 different toppings at random, find the probability that

(a) neither topping is anchovies

(b) pepperoni is one of the toppings

6. A trimotor plane has three engines - a central engine and an engine on each wing. The plane will crash only if the central engine fails and at least one of the two wing engines fails. The probability of failure during any given flight is 0.005 for the central engine and 0.008 for each of the wing engines. Assuming that the three engines operate independently, what is the probability that the plane will crash during a flight?

7. A screening test for a certain disease is prone to giving false positives or false negatives. If a patient being tested has the disease, the probability that the test indicates a (false) negative is 0.13. If the patient does not have the disease, the probability that the test indicates a (false) positive is 0.10. Assume that 3% of the patients being tested actually have the disease. Suppose that one patient is chosen at random and tested. Find the probability that

(a) this patient has the disease and tests positive

(b) this patient does not have the disease and tests positive

(c) this patient tests positive

(d) this patient has the disease given that he or she tests positive